

Nonstandard analysis of the behavior of ergodic means of dynamical systems on very big finite probability spaces.

E.I. Gordon, L.Yu. Glebsky, C.W. Henson

January 30, 2012

1 Introduction

We discuss here the behavior of ergodic means of discrete time dynamical systems on a very big finite probability space Y (discrete dynamical systems below). The G. Birkhoff Ergodic Theorem states the eventual stabilization of ergodic means of integrable functions for almost all points of the probability space. The trivial proof of this theorem for the case of finite probability spaces shows that this stabilization happens for those time intervals, whose length n exceeds significantly the cardinality $|Y|$ of Y , i.e. $\frac{n}{|Y|}$ is a very big number.

For the case of a very big number $|Y|$ we introduce a huge class of functions on Y including, for example all bounded functions, i.e. those functions, whose values are significantly less, than $|Y|$. Functions of this class are said to be S -integrable (the formula (5) below). The class of S -integrable functions is an analog of the class of integrable functions on an infinite probability space.

We show that the behavior of ergodic means of S -integrable functions demonstrates some regularity even for those intervals, whose length is comparable with $|Y|$. The ergodic means A_n and A_m on the intervals of time $\{0, \dots, m-1\} = \bar{m}$ and $\{0, \dots, n-1\} = \bar{n}$ are approximately the same if $\frac{n}{|Y|} \approx \frac{m}{|Y|} \gg 0$. It means that if we plot the points $(\frac{n}{|Y|}, A_n)$ on the coordinate plane, we obtain the graph of a function continuous on $(0, \infty)$ (Theorem 6). The behavior of this function in the neighborhood of the origin is more complicated. We show in Example 3 below the existence of an S -integrable function, for which there exist very big intervals \bar{m}, \bar{n} such that $\frac{n}{|Y|} \approx \frac{m}{|Y|} \approx 0$, but $A_n \not\approx A_m$. However, Theorem 7 shows stabilization of ergodic means on some initial segment of very big moments. In other words there exists a very big number m such that for all very big numbers $n < m$ one has $A_n \approx A_m$ for almost all $y \in Y$, i.e. the share of those $y \in Y$, for which the statement is not true, is infinitesimal. It is interesting that the proof of this theorem uses the Ergodic Theorem for infinite probability spaces and is equivalent to the last one in some sense.

We consider specially the case of discrete dynamical systems that are approximations of dynamical systems on compact metric spaces. We introduce here a definition of such approximations (Definition 5 below). The existence of approximations in the sense of Definition 5 is proved for a huge class of dynamical systems on compact metric spaces (see Section 4).

The approach to approximation suggested here differs from the most popular approach in ergodic theory based on Rokhlin's Theorem (see e.g. [3]). The Nonstandard Analysis (NSA) approach to Rokhlin's finite approximations of Lebesgue dynamical systems will be discussed in another paper. Some preliminary results were announced in [8]. Rokhlin's approximations have many interesting applications to ergodic theory, especially to problems connected with the entropy of dynamical systems. However, Definition 5 is more appropriate for investigation of computer simulation of continuous dynamical systems (see e.g. Example 6). We show also that the existence of a dynamical system on a compact metric space, for which a given very big finite dynamical system is an approximation, gives some additional information about the behavior of the finite dynamical system on very big intervals of time (see Proposition 22 and Theorem 9 below).

Our approach provides some deeper understanding of the interrelation between very big discrete dynamical systems and continuous dynamical systems in the spirit of the approach formulated in [14]: "Continuous analysis and geometry are just degenerate approximations to the discrete world... While discrete analysis is conceptually simpler ... than continuous analysis, technically it is usually much more difficult. Granted, real geometry and analysis were necessary simplifications to enable humans to make progress in science and mathematics....". In some sense, our paper contributes to this idea for dynamical systems.

Properties involved in the discussion above (very big set, very small number, etc.) obviously are not well defined. They strongly depend on the problems, where they are used. Let us call them *vague* properties. They cannot be formalized in the framework of the standard mathematics based on the G. Cantor's Set Theory. In this theory a set is understood as a collection of objects that satisfy a certain property that is well defined. This means that any two persons agree about any object, if this object has a given property or not. In other words one can definitely say about any object, if this object is an element of a given set or this is not the case.

Vague properties do not define sets. Consider, for example, the collection Ω of all very big natural numbers. If we accept the existence of a very big number, then the collection Ω is not a set. Indeed, if Ω is a set, then, obviously $\mathbb{N} \setminus \Omega$ is a set. It is clear intuitively, that $0 \in \mathbb{N} \setminus \Omega$ and if $n \in \mathbb{N} \setminus \Omega$, then $n + 1 \in \mathbb{N} \setminus \Omega$. So, by Axiom of Induction $\Omega = \emptyset$.

This argument comes up to the well-known paradox of a heap sand due to Eubulides (IV century B.C.): A single grain of sand is certainly not a heap. Nor is the addition of a single grain of sand enough to transform a non-heap into a heap: when we have a collection of grains of sand that is not a heap, then adding a single grain will not create a heap. And yet we know that at some point we will have a heap.

This paradox cannot be resolved in the framework of conventional (standard) mathematics, since the property "to be a heap of sand" is a vague property. On the other hand, vague properties are very common in natural sciences, economy and other areas of application of mathematics. Arguments, using them can be met in many investigations in these areas. These arguments seem to be quite convincing. Moreover, we will see below that the formalization in the framework of standard mathematics of some statements and arguments involving vague properties may be too complicated or even irrelevant.

Nonstandard Analysis, discovered by A. Robinson in the 60-s of the previous century introduced constant infinitesimals and infinitely large numbers in mathematics on the contemporary level of mathematical rigor. It opened the way to use vague collections (called the external sets in NSA) rigorously. The methods of NSA found numerous applications in the various areas of mathematics from mathematical physics to mathematical economics (see, e.g. [1, 9, 11]). However, in the most of the papers nonstandard analysis is used as a tool to obtain results in standard mathematics.

The results mentioned in the beginning of this Introduction have more natural formulations in terms of vague properties rather than in the framework of the standard mathematics. Some of them, like Theorem 6, can be simply reformulated in the framework of standard mathematics in terms of sequences of finite probability spaces, other, like Theorem 7, do not have simple meaningful standard formulation. However, Theorem 7 has clear meaningful sense and can even be monitored in computer experiments (see Example 3).

Section 2 contains a brief introduction to NSA. We discuss the formalization of the vague properties mentioned above. In particular, the formalization of the notion of a very big (very small) number is the formal definition of *infinite* (*infinitesimal*) numbers in the NSA. The numbers that are not very big are called *bounded* or *finite*. We say that two elements α and β of a metric space are infinitesimally close ($\alpha \approx \beta$), if the distance between them is infinitesimal. A very big finite set is defined as a set, whose cardinality is an infinite natural number. Not very big sets are said to be standardly finite. As a rule we call them just finite sets if it does not yield a misunderstanding. There are no new results in this section, however, the exposition of the introduction to NSA is new. Some proofs in this section are given for illustration of the basic principles of NSA.

We try to make the exposition in Section 2 not too formal. Nonstandard analysis has one feature, that makes the achievement of this goal a little bit more difficult, than in the standard mathematics. One of the main axioms of NSA, the *Transfer principle* is a statement about all conventional mathematical propositions. To make this statement mathematically rigorous one has to provide a formal definition of mathematical language. This can be done, for example, in the framework of the Axiomatic Set Theory. To avoid excessive formalization, we skip formal description of the mathematical language, assuming that every reader understand intuitively what is a conventional mathematical proposition and can easily apply the Transfer Principle to any concrete proposition.

The formulations of the main results of the paper formulated both in the language of the NSA and in the framework of the standard mathematic, their discussion, illustration by examples and proofs of some simple statements are contained in Section 3. It is possible to understand the formulations of the main theorems of the paper (Theorem 6, 7, 8 and 9) on the intuitive ("physical") level before reading Section 2, if to interpret an infinite number as a very big number, an infinitesimal as a very small number, a hyperfinite set as very big finite set, an internal set as a usual set, an external set as a collection of objects defined by a vague property. For example, under this interpretation the set ${}^*\mathbb{N}$ is the set of all natural numbers, while \mathbb{N} is a collection of not very big numbers.

The rigorous formulations of the definitions and of the main results in terms of sequences of dynamical systems

on finite probability spaces are contained in the part iv) of Section 3. The proofs of Theorems 6,7,8 are contained in Section 4.

Acknowledgements. The authors are grateful to Peter Loeb, Edgardo Ugalde for helpful discussions of various parts of this paper and to Andrew Mertz and Kamlesh Parwani for their help with computer experiments.

2 Basic Nonstandard Analysis

i) We deal with some standard universe \mathbb{S} that contains all objects necessary to develop a huge part of standard mathematics.

Definition 1 A set \mathbb{S} is said to be a standard universe, if

1. the field $\mathbb{R} \in \mathbb{S}$,
2. $a \in A \in \mathbb{S} \implies a \in \mathbb{S}$,
3. $A \in \mathbb{S} \implies \mathcal{P}(A) = \{B \subseteq A\} \in \mathbb{S}$,
4. $A, B \in \mathbb{S} \implies A \times B, A^B \in \mathbb{S}$,
5. if all elements a of $A \in \mathbb{S}$ are sets, then $\left(\bigcup_{a \in A} a\right) \in \mathbb{S}$,
6. any finite set of elements of \mathbb{S} is an element of \mathbb{S} ,

Proposition 1 1). If a set $A \in \mathbb{S}$ and $B \subseteq A$, then $B \in \mathbb{S}$

2). If $I \in \mathbb{S}$, $\{A_i \mid i \in I\} \in \mathbb{S}$, then $\prod_{i \in I} A_i \in \mathbb{S}$

Proof. 1) Since $B \in \mathcal{P}(A)$ and $\mathcal{P}(A) \in \mathbb{S}$ by property 3 of Definition 1 then $B \in \mathbb{S}$ by property 2.

2) The set $B = \prod_{i \in I} A_i \in \mathbb{S} \subseteq \left(\bigcup_{i \in I} A_i\right)^I = A$. The set $A \in \mathbb{S}$ by properties 4 and 5. Thus, $B \in \mathbb{S}$ by the statement 1 of this Proposition. \square

We use the following notation: Let \mathfrak{A} be any collection of objects and let Φ be a standard sentence. We write $\mathfrak{A} \models \Phi$ if Φ is true in \mathfrak{A} . This means that Φ is true, if all variables involved in it assume values in \mathfrak{A} . In this paper we use for \mathfrak{A} either the collection \mathbb{S} or the collection $^*\mathbb{S}$ introduced below.

It is easy to see that most part of the mathematical theorems are true in \mathbb{S} . Indeed, since the field $\mathbb{R} \in \mathbb{S}$ then the operations of addition and multiplication, as well as the order relation is in \mathbb{S} . Since the elements of all sets in \mathbb{S} are also in \mathbb{S} the axioms of linearly ordered fields for \mathbb{R} are true in \mathbb{S} . The axiom of the least upper bound for \mathbb{R} is true in \mathbb{S} due to Proposition 1 (1). One of the important axioms of set theory is the

Separation Axiom. For an arbitrary set B , standard property $\Phi(x, y_1, \dots, y_n)$ and elements t_1, \dots, t_n there exists a set $C = \{b \in B \mid \Phi(b, t_1, \dots, t_n) \text{ is true}\}$.

This axiom also follows from Proposition 1 (1): if $B \in \mathbb{S}$ and $t_1, \dots, t_n \in \mathbb{S}$, then $C \in \mathbb{S}$, since $C \subseteq B$.

The *Axiom of Choice* states that the direct product of any family of non-empty sets is a non-empty set. Its validity in \mathbb{S} follows from Proposition 1 (2) and Definition 1 (1).

It is accepted by the most part of mathematicians that any mathematical statement can be formalized and proved (if it is provable) in the framework of the Zermelo-Fraenkel axiomatics for set theory (**ZFC**). Besides the Separation Axiom and the Axiom of Choice this system contains the Axiom of infinity that is true in \mathbb{S} , since $\mathbb{N} \in \mathbb{S}$, the Axiom of the Unordered Pair that follows from Definition 1 (6), the Axiom of Union, the Axiom of the Power Set that follows from Definition 1 (3), the Axiom of Regularity¹ and the Axiom of Replacement, that is not true in \mathbb{S} . The axiomatics that contains all listed above axioms except the Axiom of Replacement is the Zermelo axiomatics. The Zermelo Axiomatic is enough for formalization of all concrete mathematics (analysis, differential equations, mathematical physics, geometry, etc.), while the Replacement Axiom is used only for the needs of Foundations of Mathematics. For example, it is used in the proof of existence of a set \mathbb{S} , that satisfies Definition 1.

The above discussion justifies the following

¹ Actually the axiom of regularity was introduced later by John von Neumann

Metatheorem 1. Every theorem provable in Zermelo Axiomatics is true in \mathbb{S} .

ii) We extend the standard universe \mathbb{S} by adding infinite, numbers, infinitesimals and some other objects. A good intuition for working with the nonstandard extension $^*\mathbb{S}$ of \mathbb{S} is provided by the following point of view. We consider the standard universe \mathbb{S} as the universe of visual objects, while $^*\mathbb{S}$ is obtained by adding to \mathbb{S} objects visual through a microscope (e.g. infinitesimals) and through a telescope (e.g. infinite numbers).

If $t \in \mathbb{S}$ and t is not a set, then $t \in ^*\mathbb{S}$. If a set $A \in \mathbb{S}$, then A may be extended in $^*\mathbb{S}$, by adding some nonstandard elements. The nonstandard extension of a set $A \in \mathbb{S}$ is denoted by *A . The set $^*A \in ^*\mathbb{S}$. For example, we will see later that the nonstandard extension $^*\mathbb{R}$ of the set \mathbb{R} consists of infinite numbers, infinitesimals and numbers of the type $t + \alpha$, where $t \in \mathbb{R}$ and $\alpha \approx 0$. The elements of \mathbb{S} , that are not sets and the sets of the form *A , where $A \in \mathbb{S}$, are said to be **standard** elements of $^*\mathbb{S}$. To study the universe $^*\mathbb{S}$ we use the conventional language of mathematics extended by the predicate $\mathbf{S}(x)$ that is interpreted as " x is standard". We denote by \mathbf{S} also the collection of all standard elements of $^*\mathbb{S}$. So, to write $\mathbf{S}(x)$ is the same as to write $x \in \mathbf{S}$. We use the abbreviations $\forall^{\text{st}} x \dots$ and $\exists^{\text{st}} x \dots$ for $\forall x(\mathbf{S}(x) \implies \dots)$ and $\exists x(\mathbf{S}(x) \& \dots)$ respectively. Let Φ be a proposition that may contain some free variables assuming values in \mathbb{S} or in $^*\mathbb{S}$. Then Φ^{st} is a proposition that is obtained from Φ by replacing any quantifier \forall (\exists) by \forall^{st} (\exists^{st}). All elements of $^*\mathbb{S}$ (sets and not sets) are said to be internal elements. Propositions formulated in conventional language are said to be *internal*. Propositions containing the predicate \mathbf{S} and the map * are said to be *external*. External propositions are used to describe vague properties discussed in the Introduction.

We introduce now the axioms for the nonstandard universe $^*\mathbb{S}$. These axioms are simplified versions of the axioms of one the axiomatic nonstandard set theories - the theory **HST** (Hrbacek Set Theory) ([10]).

We say that a proposition is a *sentence* if all variables involved in Φ are connected by quantifiers

I. There exist an injective map $^*: \mathbb{S} \rightarrow \mathbf{S}$ ($^*(\mathbb{S}) := \mathbf{S}$) such that for any element $t \in \mathbb{S}$ that is not a set one has $^*t = t$.

II. (Transitivity of $^*\mathbb{S}$) If $A \in ^*\mathbb{S}$ and $a \in A$, then $a \in ^*\mathbb{S}$.

III. (Transfer Principle) If $\Phi(x_1, \dots, x_n)$ is an internal proposition and $a_1, \dots, a_n \in \mathbb{S}$, then

$$\mathbb{S} \models \Phi(a_1, \dots, a_n) \iff ^*\mathbb{S} \models \Phi(^*a_1, \dots, ^*a_n) \iff ^*\mathbb{S} \models \Phi^{\text{st}}(^*a_1, \dots, ^*\alpha_n).$$

The Transfer Principle immediately implies

Metatheorem 2. Every theorem provable in Zermelo Axiomatics is true in $^*\mathbb{S}$.

The next propositions easily follow from the Transfer Principle.

Proposition 2 *The bijection * preserves the boolean operations on sets and finite cartesian products.*

Proof. Let $A, B, t \in \mathbb{S}$. Then one has $t \in (A \cap B) \iff t \in A \& t \in B$. So, by Transfer Principle

$$^*t \in ^*(A \cap B) \iff ^*t \in ^*A \& ^*t \in ^*B.$$

Thus,

$$\forall^{\text{st}} t (t \in ^*(A \cap B) \iff t \in ^*A \& t \in ^*B).$$

Again, by Transfer Principle

$$\forall t (t \in ^*(A \cap B) \iff t \in ^*A \& t \in ^*B)$$

is true, which means that $^*(A \cap B) = ^*A \cap ^*B$. For the other operations the proof is similar. \square

Remark 1 *To understand the second part of the proof, keep in mind that the set *A , *B and $^*(A \cap B)$ may contain not only standard elements.*

We say that the proposition $\Phi(x)$ defines the standard element $t \in \mathbb{S}$ if the following statement is true in \mathbb{S} :

$$\Phi(t) \& \forall y (\Phi(y) \rightarrow y = t)$$

.

Proposition 3 *If a proposition $\Phi(x)$ defines an element $t \in \mathbb{S}$, then it defines $^*t \in ^*\mathbb{S}$*

\square

Corollary 1 a). $^*\emptyset = \emptyset$.

b). If a set $A = \{\alpha_1, \dots, \alpha_n\} \in \mathbb{S}$, then $^*A = \{^*\alpha_1, \dots, ^*\alpha_n\}$.

To prove the statement b) of this Corollary first prove it for $n = 1$ using the Transfer Principle, then apply the induction by $n \in \mathbb{N}$.

IV. (Idealization Principle) If a set $A \in \mathbb{S}$ is infinite, then $^*A \setminus A \neq \emptyset$.

Proposition 4 If $N \in ^*\mathbb{N} \setminus \mathbb{N} = ^*\mathbb{N}_\infty$, then for any $n \in \mathbb{N}$ one has $N > n$.

Proof. If $N \leq n$ for some $n \in \mathbb{N}$, then $N \in \{0, \dots, n\} = \{0, \dots, n\}$ by Corollary 1 (b). Thus, $N \in \mathbb{N}$. The contradiction. \square

Obviously, if $N \in ^*\mathbb{N}_\infty$, then $N - 1 \in ^*\mathbb{N}_\infty$. Thus, the set $^*\mathbb{N}_\infty$ does not have a minimal element and the set \mathbb{N} satisfies the antecedent of Induction Principle, however $\mathbb{N} \neq ^*\mathbb{N}$. So, the sets \mathbb{N} and $^*\mathbb{N}_\infty$ are not internal sets, since, by the Transfer Principle, the Induction Principle is applicable to internal subsets of $^*\mathbb{N}$.

We see, thus, that the property 3 of Definition 1 fails for the nonstandard universe $^*\mathbb{S}$. There exists a set A and a subset $B \subseteq A$, such that $A \in ^*\mathbb{S}$ and $B \notin ^*\mathbb{S}$.

Definition 2 We say that a set B is external, if it is not an internal set, but is a subset of an internal set.

We extend the nonstandard universe by adding all external sets: $\tilde{\mathbb{S}} := ^*\mathbb{S} \cup \{B \subseteq ^*\mathbb{S} \mid B \text{ is an external set}\}$

We use the abbreviations $\forall^{\text{int}} x \dots$ and $\exists^{\text{int}} x \dots$ for $\forall x (x \in ^*\mathbb{S} \implies \dots)$ and $\exists x x \in ^*\mathbb{S} \& \dots$ respectively. Let Φ be a proposition (maybe external) that may contain some free variables assuming values in \mathbb{S} or in $^*\mathbb{S}$. Then Φ^{int} is a proposition that is obtained from Φ by replacing any quantifier \forall (\exists) by \forall^{int} (\exists^{int}). There is no need to write $\forall^{\text{int}^{\text{st}}} x$ or $\exists^{\text{int}^{\text{st}}} x$, since if $x \in \mathbb{S} \implies x \in ^*\mathbb{S}$. Notice that Φ^{int} is an external proposition even if Φ is an internal proposition since the proposition $x \in ^*\mathbb{S}$ is defined in terms of the map * . Obviously for every external proposition Φ and every internal proposition Ψ one has

$$\tilde{\mathbb{S}} \models \Phi^{\text{int}} \iff ^*\mathbb{S} \models \Phi, \quad \tilde{\mathbb{S}} \models \Psi^{\text{st}} \iff \mathbb{S} \models \Psi. \quad (1)$$

Remark 2 The propositions of the form Φ^{int} we call in this paper Nelson-type propositions, since E. Nelson was the first who suggested a formal axiomatic (IST) for internal sets in conventional language extended by the predicate "x is standard" - the analog of our predicate $x \in \mathbb{S}$ and who wrote the first exposition of probability theory in the framework of IST [12, 13]

The main results of this paper are formulated as Nelson-type sentences, since only this sentences are most intuitively clear formalizations of the statements containing the vague notions discussed above. However, in the proves we use propositions that involve variables assuming values of arbitrary external sets. These propositions make many of proofs much simpler, than if we restrict ourselves to the arguments that can be formalized in IST.

Definition 3 We say that an internal set A is finite, if there exists $n \in ^*\mathbb{N}$ and an internal bijection $\phi : A \rightarrow \{1, \dots, n\}$. In this case we say that the cardinality $|A|$ of A is equal to n . If $|A| \in ^*\mathbb{N}_\infty$, then we say that A is a hyperfinite set. If $|A| \in \mathbb{N}$, then we say that A is standardly finite (s-finite) set.

Remark 3 If Φ is the definition of a finite set in the conventional mathematics, then the statement used for the definition of a finite set in Definition 3 is the proposition Φ^{int} . So it would be more correct to call the number $|A|$ defined in Definition 3 the internal cardinality. However, the real cardinality of A , i.e. the cardinality of A in the "global" universe of all sets may strongly depend on the properties of $^*\mathbb{S}$ and is never used in applications of NSA. So, we prefer to keep the term "cardinality" for the internal cardinality, and to call the real cardinality of a set the external cardinality. It agrees with our intuition, according to which the set $^*\mathbb{N}$ is the set of all natural numbers that includes also those numbers that can be seen through a telescope. The cardinality should be a well-defined (not vague) notion, that is why it must be an internal (not external) set. The definition of a hyperfinite set is a formalization of the vague notion of a very big set. The definition of an s-finite set is a formalization of the vague notion of a not too big set. Obviously the external cardinality of a hyperfinite set is infinite. On the other hand, it can be easily proved by induction that if an internal set is s-finite, then its internal cardinality is equal to its external cardinality and every set, whose external cardinality is a standard natural number, is an internal set.

V. (Saturation Principle)². If an external sequence $\{A_n \mid n \in \mathbb{N}\}$ of internal sets has a finite intersection property (i.e. $\forall n \in \mathbb{N} \bigcap_{k \leq n} A_k \neq \emptyset$), then $\bigcap_{n \in {}^*\mathbb{N}_\infty} A_n \neq \emptyset$.

iii) The following definition contains the formalization of the vague notions of very big and very small numbers.

Definition 4 We say that

1. a number $\Omega \in {}^*\mathbb{R}$ is infinite ($\Omega \sim \infty$), if $|\Omega| > N$ for all $N \in \mathbb{N}$. A number $\alpha \in {}^*\mathbb{R}$ that is non-infinite is said to be bounded or finite ($\alpha \ll \infty$).
2. A number $\alpha \in {}^*\mathbb{R}$ is said to be infinitesimal $\alpha \approx 0$, if $|\alpha| < \frac{1}{N}$ for all $N \in \mathbb{N}$. Two numbers α and β are infinitesimally close $\alpha \approx \beta$, if $\alpha - \beta \approx 0$. We write $|a| \gg 0$, if α is not an infinitesimal number.
3. A number $t \in \mathbb{R}$ is said to be a standard part (or a shadow) of a bounded number α ($t = {}^\circ\alpha$), if $t \approx \alpha$.

The existence of infinite and infinitesimal numbers follows from Proposition 4

We denote the set of all bounded numbers by ${}^*\mathbb{R}_b$. If $t \in \mathbb{R}$, then the set $M(t) = \{\alpha \in {}^*\mathbb{R} \mid \alpha \approx t\}$ is called the monad of t .

The properties of infinite, bounded and infinitesimal numbers are similar to the properties of sequences that diverge to infinity, are bounded and tend to 0 respectively in standard calculus. They can be summarized as follows.

Proposition 5 1) $\Omega \sim \infty \iff \Omega^{-1} \approx 0$.

2) The set ${}^*\mathbb{R}_b$ is a subring of the field ${}^*\mathbb{R}$ and the set $M(0)$ is an ideal in the ring ${}^*\mathbb{R}_b$.

We leave a simple proof of this proposition as an exercise.

Theorem 1 1) Every $\alpha \in {}^*\mathbb{R}_b$ has a unique standard part.

2) The map ${}^\circ: {}^*\mathbb{R}_b \rightarrow \mathbb{R}$ is a homomorphism of a ring ${}^*\mathbb{R}_b$ onto the field \mathbb{R} .

Proof. The only non-trivial statement is the existence of standard part for any bounded element. Let $\alpha \in {}^*\mathbb{R}_b$ be bounded. Then there exists $s \in \mathbb{R}$ such that $s > \alpha$. Consider the set $P = \{p \in \mathbb{R} \mid p < \alpha\} \subseteq \mathbb{R}$. This set is nonempty since α is bounded not only from above, but also from below. Thus, there exists $t = \sup P$ in \mathbb{R} . If $t < \alpha$, then $t + \frac{1}{n} < \alpha$ for any $n \in \mathbb{N}$. Otherwise $\sup P \geq t + \frac{1}{n}$ for some $n \in \mathbb{N}$. So, $\alpha - t < \frac{1}{n}$ for any $n \in \mathbb{N}$, i.e. $\alpha - t \approx 0$. If $t > \alpha$ the proof is similar. \square

Let $(X, \rho) \in \mathbb{S}$ be a metric space. In what follows we write this and similar sentences as "Let (X, ρ) be a standard metric space". Then by the Transfer Principle ${}^*\mathbb{S} \models ({}^*X, {}^*\rho)$ is a metric space. In what follows for any proposition Φ instead of writing ${}^*\mathbb{S} \models \Phi$ we write " Φ in ${}^*\mathbb{S}$ ". For example, the previous statement may be written as $({}^*X, {}^*\rho)$ is a metric space in ${}^*\mathbb{S}$.

For any $\xi_1, \xi_2 \in {}^*X$ we write $\xi_1 \approx \xi_2$, if ${}^*\rho(\xi_1, \xi_2) \approx 0$. For $x \in X$ and $\xi \in {}^*X$ we write $(x = \text{st}(\xi))$, if $x \approx \xi$. We say in this case that x is a standard part of ξ ³. Obviously, $\text{st}(\xi)$ is defined uniquely. An element of $\xi \in {}^*X$ is said to be nearstandard, $\text{st}(\xi)$ exists. In particular, ${}^*\mathbb{R}_b$ is the set of all nearstandard elements of ${}^*\mathbb{R}$. Similarly, the set of all nearstandard elements of *X is denoted by *X_b . For $x \in X$ the set $\{\xi \in {}^*X \mid \xi \approx x\}$ is said to be a monad of x and denoted by $M(x)$. For an arbitrary $0 < \varepsilon$ and $x \in X$ let $B_\varepsilon(x) = \{\xi \in X \mid \rho(\xi, x) \leq \varepsilon\}$. Then it is easy to see that

$$M(x) = \bigcap_{n \in \mathbb{N}} {}^*B_{\frac{1}{n}}(x). \quad (2)$$

The simple proof of the following proposition can be found in the books [1, 11, 9].

Proposition 6 Let (X, ρ) be a standard separable metric space and $A \subseteq X$. Then the following statements are true⁴.

1. The set A is open if and only if $\forall x \in A \ M(x) \subseteq {}^*A$.
2. The set A is closed if and only if $\forall x \in X \ (M(x) \cap {}^*A \neq \emptyset \implies x \in A)$.

²We introduce here the weakest form of the Saturation Principle. However, this form is enough for our goals.

³For the case of the metric space \mathbb{R} we use also the notation ${}^\circ$ for the standard part.

⁴These statements are true without the assumption of separability of X , if the nonstandard universe *X satisfies some stronger Saturation Principle.

3. The set A is compact if and only if it is closed and every element $\xi \in {}^*A$ is nearstandard. In particular, X is a compact metric space, if and only if every element $\xi \in {}^*X$ is nearstandard.

Proof. To illustrate how the axioms I-V work let us prove that every X compact metric space satisfies the second statement of 3). Suppose that X is a compact metric space. Then for every $n \in \mathbb{N}$ there exists a finite set $\{x_1, \dots, x_k\} \subseteq X$ such that $X = \bigcup_{i=1}^k B_{\frac{1}{n}}(x_i)$. Then, by the Transfer Principle ${}^*X = \bigcup_{i=1}^k {}^*B_{\frac{1}{n}}(x_i)$. In other words, $\forall \xi \in {}^*X \forall n \in \mathbb{N} \exists \text{st}_X \xi \in {}^*B_{\frac{1}{n}}(x)$ (A). Suppose that there exists $\xi \in {}^*X$ that is not nearstandard. It means that $\forall \text{st}_X \xi \notin M(x)$, i.e. by formula (2) $\xi \notin \bigcap_{n \in \mathbb{N}} {}^*B_{\frac{1}{n}}(x)$. Then, by the Saturation Principle there exists $n \in {}^*\mathbb{N}_\infty$ such that $\xi \notin {}^*B_{\frac{1}{n}}(x)$, which contradicts (A). \square

iii) This part of Section 2 contains some well-known facts of nonstandard analysis that are necessary only for the proofs of results formulated in Section 3. These proofs are contained in Section 4.

Theorem 2 Let $A \subseteq {}^*\mathbb{R}$ be an internal set.

1. If $\mathbb{N} \subseteq A$, then $\{0, 1, \dots, N\} \subseteq A$ for some $N \in {}^*\mathbb{N}_\infty$.
2. If ${}^*\mathbb{N}_\infty \subseteq A$, then ${}^*\mathbb{N} \setminus \{0, 1, \dots, n\} \subseteq A$ for some $N \in \mathbb{N}$.
3. If $M(0) \subseteq A$, then $(-t, t) \subseteq A$ for some $0 < t \in \mathbb{R}$.
4. (Robinson's Lemma) Let $\langle s_n \mid n \in {}^*\mathbb{N} \rangle$ be an internal sequence such that $s_n \approx 0$ for all $n \in \mathbb{N}$. Then there exists $N \in {}^*\mathbb{N}_\infty$ such that $s_n \approx 0$ for all $n < N$.
5. Let $\langle N_n \mid n \in \mathbb{N} \rangle$ be an external sequence of infinite numbers. Then there exists $N \in {}^*\mathbb{N}_\infty$ such that $N < N_n$ for all $n \in \mathbb{N}$.

Proofs of the statements 1-4 can be found e.g. in [11], p. 53.

Proof of the statement 5. Consider an external sequence of internal sets $B_n \langle \{n, \dots, N_n\} \mid n \in \mathbb{N} \rangle$. This sequence has obviously a finite intersection property. By the Saturation Principle $\bigcap_{n \in \mathbb{N}} B_n \neq \emptyset$. Any element of this intersection satisfies the conditions of the statement 5. \square

We say that a set $I \subseteq {}^*\mathbb{N}_\infty$ is an *initial segment of infinite numbers* if $I = \{0, \dots, N\} \cap {}^*\mathbb{N}_\infty$.

Theorem 3 Let $\langle a_n \mid n \in {}^*\mathbb{N} \rangle$ be an internal sequence of nonstandard real numbers (elements of ${}^*\mathbb{R}$). Then $\lim_{n \rightarrow \infty} {}^\circ a_n = a \in \mathbb{R}$ if and only if $a_L \approx a$ for all L in some initial segment of infinite numbers.

Proof. Longrightarrow Let $\lim_{n \rightarrow \infty} {}^\circ a_n = a \in \mathbb{R}$. For any $m \in \mathbb{N}$ there exists $n_m \in \mathbb{N}$ such that the internal set $A_m = \{k \in {}^*\mathbb{N} \mid |a_k - a| \leq \frac{1}{m}\}$ contains all $k \in \mathbb{N}$ such that $k \geq n_m$. Then, by Theorem 2 (1) there exist $N_m \in {}^*\mathbb{N}_\infty$ such that $\{n_m, \dots, N_m\} \subset A_m$. By Theorem 2 (1) there exists $N \in {}^*\mathbb{N}_\infty$ such that $N < N_m$ for all $m \in \mathbb{N}$. This N satisfies the conditions of the theorem.

← Let $N \in {}^*\mathbb{N}_\infty$ be such that $a_L \approx a$ for all $L \leq N$, $L \in {}^*\mathbb{N}_\infty$. Fix an arbitrary $m \in \mathbb{N}$. Then the internal set $B = \{k \in {}^*\mathbb{N} \mid |a_k - a| < \frac{1}{m}\} \supseteq \{L \in {}^*\mathbb{N}_\infty \mid L \leq N\}$. Thus, by Theorem 2 (2) there exists $n \in \mathbb{N}$ such that $B \supseteq \{n, \dots, N\} \supseteq \{k \in \mathbb{N} \mid k > n\}$. So, $\forall m \in \mathbb{N} \exists n \in \mathbb{N} \forall k \in \mathbb{N} k > n \implies |{}^\circ a_k - a| < \frac{1}{m}$. This means that $\lim_{k \rightarrow \infty} {}^\circ a_k = a$. \square

Remark 4 Notice, that the sequence $\langle {}^\circ a_n \mid n \in \mathbb{N} \rangle \in \mathbb{S}$ is a standard sequence and, thus, the sequence $\langle {}^{**}a_n \mid n \in {}^*\mathbb{N} \rangle$ is defined. This sequence is not necessarily equal to the initial internal sequence $\langle a_n \mid n \in {}^*\mathbb{N} \rangle$. The only statement that can be claimed is that the entrees of these two sequences are infinitesimally close on an interval $\{0, \dots, N\}$ for some $N \in {}^*\mathbb{N}_\infty$.

Let X be a compact metric space, $Y \subseteq {}^*X$. We say that Y is a *dense* subset of *X , if $\forall x \in X \exists y \in Y x \approx y$. Proposition 6 implies that the last statement is equivalent to the statement $\forall x \in {}^*X \exists y \in Y x \approx y$.

If $Y \subseteq X$ and X is a metric space, then we say that a function $f : Y \rightarrow {}^*\mathbb{R}$ is *S-continuous* on Y , if $\forall y_1, y_2 \in Y y_1 \approx y_2 \implies f(y_1) \approx f(y_2)$.

Theorem 4 Let (X, ρ) , (Z, d) be standard separable metric spaces, X be a compact space and $Y \subseteq {}^*X$ be a dense internal subset of *X .

1) A function $f : X \rightarrow Z$ is continuous if and only if ${}^*f : {}^*X \rightarrow {}^*Z$ is S -continuous on *X .

2) Let $F : Y \rightarrow {}^*Z_b$ be an internal function that is S -continuous on Y , then the function $f : X \rightarrow Z$ defined by the formula $f(st(\xi)) = st(F(\xi))$ is a continuous function.

In what follows the function f defined in the statement 2) of the theorem is said to be the *visual image* of F , if $F \subseteq {}^*\mathbb{R} \times {}^*\mathbb{R}$. More generally, if $A \subseteq {}^*\mathbb{R} \times {}^*R$, the the set ${}^\circ A = \{({}^\circ a, {}^\circ b) \mid (a, b) \in A\}$ is said to be the visual image of A . This definition is specific for this paper. Usually, in NSA the set ${}^\circ A$ is said to be the shadow of A .

Proof. We prove the statement \implies for 1) and the statement \impliedby for 2).

1) \implies . Since X is a compact space and f is continuous on X , then f is uniformly continuous on X . This means that $\forall \varepsilon \exists \delta, \forall x_1, x_2 \in X \rho(x_1, x_2) < \delta \implies d(f(x_1), f(x_2)) < \varepsilon$ is true in \mathbb{S} . By the Transfer Principle the statement

$$\forall^{st} \varepsilon \exists^{st} \delta, \forall \xi_1, \xi_2 \in {}^*X {}^*\rho(\xi_1, \xi_2) < \delta \implies {}^*d(f(\xi_1), f(\xi_2)) < \varepsilon \quad (3)$$

is true in ${}^*\mathbb{S}$. If $\xi_1 \approx \xi_2$, then the antecedent of the implication in the statement (3) is true for any standard δ . Thus, the consequence of this implication is true for any standard $\varepsilon > 0$. This means that ${}^*f(\xi_1) \approx {}^*f(\xi_2)$.

2) \impliedby . Due to Proposition 6 (3) the function f is defined correctly. We have to prove that f is uniformly continuous on X . By the definition of f it is enough to prove that

$$\forall^{st} \varepsilon > 0 \exists^{st} \delta > 0 \forall \xi_1, \xi_2 \in Y {}^*\rho(\xi_1, \xi_2) < \delta \implies {}^*d(F(\xi_1), F(\xi_2)) < \varepsilon. \quad (4)$$

Fix an arbitrary standard $\varepsilon > 0$. Due to the S -continuity of F , the internal set

$$B = \{0 < \delta \in {}^*\mathbb{R} \mid \forall \xi_1, \xi_2 \in Y {}^*\rho(\xi_1, \xi_2) < \delta \implies {}^*d(F(\xi_1), F(\xi_2)) < \varepsilon\}$$

contains all $0 < \delta \approx 0$. Thus, by Theorem 2 (3), there exist a $\delta \in \mathbb{S}$ such that ${}^*(0, \delta) \subseteq B$. This proves the statement (4). \square

iv) We list now the necessary definitions and facts concerning Loeb spaces. We need here only a particular case of a Loeb space, namely the Loeb space constructed from the hyperfinite set Y endowed with the uniform probability measure.

Define the internal finitely additive measure μ on the algebra $\mathcal{P}^{int}(Y)$ of internal subsets of Y by the formula

$$\mu(B) = \frac{|B|}{M}.$$

This measure induces the external finite additive measure ${}^\circ\mu$ on $\mathcal{P}^{int}(Y)$

The Saturation Principle and the Caratheodory Theorem imply the possibility to extend ${}^\circ\mu$ on the σ -algebra $\sigma(\mathcal{P}^{int}(Y))$ generated by $\mathcal{P}^{int}(Y)$. The Loeb space with the underlying set Y is the probability space $(Y, P_L(Y), \mu_L)$, where $P_L(Y)$ is the completion of $\sigma(\mathcal{P}^{int}(Y))$ with respect to the extension of ${}^\circ\mu$ and μ_L is the extension of ${}^\circ\mu$ on $P_L(Y)$. The measure μ_L is said to be the *Loeb measure* on Y . If necessary we use the notation μ_L^Y . We need the following property of the Loeb measure that follows immediately from the Saturation Principle.

Proposition 7 For every set $A \in \mathcal{P}_L$ there exists an internal set $B \subseteq Y$ such that $\mu_L(A \Delta B) = 0$.

Corollary 2 If $A \in \mathcal{P}_L$, then

$$\mu_L(A) = 1 \iff \forall^{st} \varepsilon > 0 \exists B \in \mathcal{P}^{int}(Y) (B \subseteq A \wedge \mu(B) > 1 - \varepsilon)$$

Remark 5 The proposition in the right hand side of this corollary is a Nelson-type proposition that will be used as a formalization of the notion "almost everywhere in Y " ("for almost all $y \in Y$ ").

For an arbitrary complete separable metric space R and an external function $f : Y \rightarrow R$ an internal function $F : Y \rightarrow {}^*R$ is said to be a *lifting* of f if $\mu_L(\{y \in Y \mid F(y) \approx f(y)\}) = 1$.

Proposition 8 A function $f : Y \rightarrow R$ is measurable iff it has a lifting.

An internal function $F : Y \rightarrow {}^*\mathbb{R}$ is said to be *S-integrable* if for all $K \in {}^*\mathbb{N}_\infty$ one has

$$\frac{1}{M} \sum_{\{y \in Y \mid |F(y)| > K\}} |F(y)| \approx 0. \quad (5)$$

We need the following properties of *S-integrable* functions.

Proposition 9 1) An *S-integrable* function is almost everywhere bounded.

2) An internal function $F : Y \rightarrow {}^*\mathbb{R}$ is *S-integrable* iff $\text{Av}(|F|) = \frac{1}{M} \sum_{y \in Y} |F(y)|$ is bounded and $\frac{1}{M} \sum_{y \in A} |F(y)| \approx 0$ for every internal $A \subseteq Y$ such that $\frac{|A|}{M} \approx 0$.

3) An external function $f : Y \rightarrow \mathbb{R}$ is integrable w.r.t. the Loeb measure μ_L iff it has an *S-integrable* lifting F , in which case

$$\int_Y f d\mu_L = {}^\circ\text{Av}(F).$$

We address readers to [1, 11] for the proofs of Propositions 7, 8 and 9.

3 Formulation and Discussion of results

i). In the sake of convenience of the references we recall the formulation of classical G. Birkhoff Ergodic Theorem. (see e.g [3, 2]).

Theorem 5 Let (X, Σ, ν) be a probability space and $T : X \rightarrow X$ a measure preserving transformation and $f \in L_1(X)$. Denote by

$$A_k(f, T, x) = \frac{1}{k} \sum_{i=0}^{k-1} f(T^i x).$$

Then

- 1) there exists the function $\hat{f}(x) \in L_1(X)$ such that $A_k(f, T, x) \rightarrow \hat{f}(x)$ as $k \rightarrow \infty$ a.e.;
- 2) the function \hat{f} is *T*-invariant, i.e. $\hat{f}(Tx) = \hat{f}(x)$ for almost all $x \in X$;
- 3) $\int_X f d\nu = \int_X \hat{f} d\nu$.

If Y is a finite set, $|Y| = M$, then every function on Y is integrable, whatever Σ and ν are. We restrict ourselves to the case of the uniform measure: $\mu(A) = \frac{|A|}{M}$ for any set $A \subseteq Y$. Then any measure preserving transformation $T : Y \rightarrow Y$ is a bijection and the integral of a function $F : Y \rightarrow {}^*\mathbb{R}$ is the average $\text{Av}(F) = \frac{1}{M} \sum_{y \in Y} F(y)$ of F . Theorem 5 is proved very easily in this case. We reformulate it as a statement in the nonstandard universe ${}^*\mathbb{S}$ assuming that F is an internal function, Y is a hyperfinite set and, thus, $M \sim \infty$.

For any $y \in Y$ denote the *T*-orbit of y by $\text{Orb}(y)$ and the period of y by $p(y)$.

Proposition 10 For any $y \in Y$, if $n \gg p(y)$, then $A_n(F, T, y) \approx \hat{F}(y)$, where

$$\hat{F}(y) = \frac{1}{|\text{Orb}(y)|} \sum_{z \in \text{Orb}(y)} F(z)$$

Corollary 3 If $n \gg M$, and T is a cycle of length M , then

$$\forall y \in Y \hat{f}(f) \approx \frac{1}{M} \sum_{y \in X} f(y) = \text{Av}(f)$$

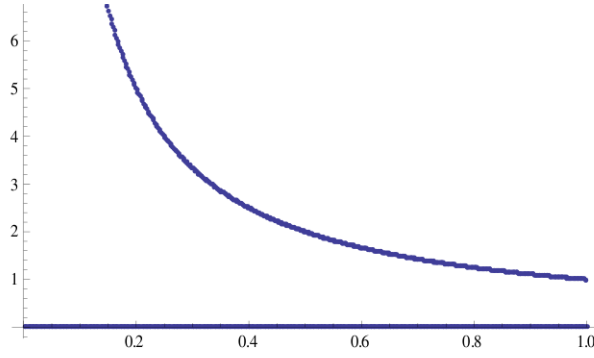


Figure 1: $\Gamma(F)$ for F defined in Eq.6

We leave a simple proof of this proposition as an exercise (see also the proof of Theorem 6 below).

ii). For the case of $M \sim \infty$ it is interesting to study the behavior of ergodic means for $n \sim \infty$ but such that $\frac{n}{M} \ll \infty$. We start with simple examples.

Example 1. Let $Y = \{0, \dots, M-1\}$, $T : Y \rightarrow Y$ is defined by the formula $T(y) = y + 1 \pmod{M}$ for any $y \in Y$. Consider the function $F : Y \rightarrow {}^*\mathbb{R}$ such that

$$F(y) = \begin{cases} M, & \text{if } y \text{ is even,} \\ -M, & \text{if } y \text{ is odd.} \end{cases} \quad (6)$$

Then for any $y \in Y$ one has

$$A_n(F, T, y) = \frac{M}{n} \cdot \begin{cases} 0, & \text{if } n \text{ is even, } y \text{ is any,} \\ 1, & \text{if } n \text{ is odd, } y \text{ is even} \\ -1, & \text{if } n \text{ is odd, } y \text{ is odd.} \end{cases} \quad (7)$$

Let us plot the set of points $\{(n/M, A_n(F, T, x)) \mid n = 0, \dots, kM\}$ for any chosen randomly $y \in Y$. The first question here is how to choose an infinite number M . Recall that the notion of an infinite number is a formalization of a notion of a very big number. Certainly, the property "to be very big" depends on the problem. Sometimes very moderate numbers can be considered as very big. In this example we consider a number M to be very big, if we see the the set of points $\{n/M \mid n = 0, \dots, M\}$ as a continuous segment. On Fig.1 $M = 1000$ and the randomly chosen $y = 698$.

We see on this picture the graphs of two functions $y = 0$ and $y = \frac{1}{x}$. The first one is the visual image of of the set of points $A = \{(n/M, A_n(F, T, x)) \mid n < M, n \text{ is even}\}$, the second one is the visual image of of the set of points $B = \{(n/M, A_n(F, T, x)) \mid n < M, n \text{ is odd}\}$.

Indeed, the set A is an internal function, $\text{dom}(A) = D_1 = \{n/M \mid n < M, n \text{ is even}\}$ and $A(n/M) = 0$. The set D_1 is dense in $[0, 1]$ and the function A is obviously S -continuous on D_1 . By Theorem 4, the function A defines the continuous function f . In this case obviously $f(t) \equiv 0$.

The set B is an internal function, whose domain $D_2 = \{n/M \mid n < M, n \text{ is odd}\}$ and $B(n/M) = \frac{1}{n/M}$. The set D_2 is also dense in ${}^*[0, 1]$, but B does not satisfy conditions of Theorem 4, since $B(n/M) \sim \infty$ as $n/m \approx 0$. However, for any $0 < a \in \mathbb{R}$ the function B restricted to the set $D_2 \cap {}^*[a, 1]$ is S -continuous. Obviously, its visual image is the function $f(t) = 1/t$ restricted to the interval $[a, 1]$. On Fig.1 $a = 0.15$

In what follows we denote the set $\{(n/M, A_n(F, T, y)) \mid 0 < n/M \leq kM\}$ by $\Gamma_k(F)$. We write $\Gamma(F)$, if $k = 1$.

It is natural to ask oneself the following question.

Under what conditions on an internal function $F : Y \rightarrow {}^*\mathbb{R}$ the visual image of the set $\Gamma_k(F)$ is the graph of a continuous function for any $k \ll \infty$ and for almost all $y \in Y$?

In view of the above discussion this question can be reformulated as follows.

Question 1 Under what conditions on a function $F : Y \rightarrow {}^*\mathbb{R}$ the ergodic means $A_n(F, T, y)$ satisfy the following property:

$$\frac{n}{M} \approx \frac{m}{M} \implies A_n(F, T, y) \approx A_m(F, T, y) \quad (8)$$

for almost all $y \in Y$?

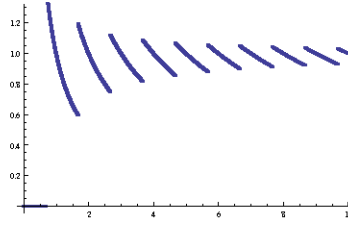


Figure 2: $\Gamma_{10}(f)$

In investigation of this question we restrict ourselves to the case, when a permutation $T : X \rightarrow X$ is a cycle of the length M . The general case can be easily reduced to this one.

Due to Corollary 3, the implication (8) holds for every function $F : Y \rightarrow {}^*\mathbb{R}$ and every $y \in Y$ if $\frac{n}{M}, \frac{m}{M} \sim \infty$. So, it is enough to consider the case of $\frac{n}{M}, \frac{m}{M} \ll \infty$.

In the following computer experiments, we illustrate that a proposition Φ holds for almost all $y \in Y$, by checking that this property holds for a randomly chosen $y \in Y$, using computer generator of random elements. As in Example 1 we use a concrete very big finite set Y , that can be considered as hyperfinite one in our problem.

The property of S -integrability of a function on a very big finite space is an analog of the property of integrability of functions on infinite probability spaces. It is easy to see that any bounded function F ($\max_{y \in Y} |F(y)| \ll \infty$) is S -integrable. The δ -function gives an example of a function F with bounded $Av(|F|)$ that is not S -integrable.

Example 2 For the same Y and T as in Example 1 consider the function $F : Y \rightarrow {}^*\mathbb{R}$ given by the formula

$$F(k) = \begin{cases} M, & k = 0 \\ 0, & k \neq 0 \end{cases} \quad (9)$$

We leave to the reader as an easy exercise to find the formula for $A_n(F, T, y)$ for this function F . On Fig.2 we show the visual image of the sets $\Gamma_{10}(f)$ for $M = 1000$, and randomly chosen $x = 322$. We see that the visual image on the first picture of Fig.2 is a graph of a function that has points of discontinuity.

Theorem 6 Let $T : Y \rightarrow Y$ be a cycle of length M . Then for every S -integrable function $F : Y \rightarrow {}^*\mathbb{R}$, for every positive $a \in \mathbb{R}$ such that $0 \ll a \ll \infty$ and for every numbers $K, L \sim \infty$ such that $\frac{K}{M} \approx \frac{L}{M} \approx a$ one has $A_K(F, T, y) \approx A_L(F, T, y)$ for all $y \in Y$.

Proof. Assume $K > L$ and estimate $|A_K(F, T, y) - A_L(F, T, y)|$. It is easy to see that

$$|A_K(F, T, y) - A_L(F, T, y)| \leq \left(\frac{1}{L} - \frac{1}{K} \right) \sum_{k=0}^{L-1} |F(T^k y)| + \frac{1}{K} \sum_{k=L}^{K-1} |F(T^k y)| = U + V.$$

One has

$$U = \left(\frac{M}{L} - \frac{M}{K} \right) \frac{1}{M} \sum_{k=0}^{L-1} |F(T^k y)| \approx 0,$$

since $\frac{M}{L} \approx \frac{M}{K} \approx \frac{1}{a}$ and $\frac{1}{M} \sum_{k=0}^{L-1} |F(T^k y)| \leq \frac{1}{M} \sum_{k=0}^{([a]+1)M-1} |F(T^k y)| = [a] Av(|F|)$ which is bounded due to the S -integrability of F .

Let $B = \{T^k y \mid k = L, \dots, K-1\}$. Then $\frac{|B|}{M} = \frac{K-L}{M} \approx 0$. Thus, $\frac{1}{M} \sum_{y \in B} |F(y)| \approx 0$, due to the S -integrability of F .

So, $V = \frac{M}{K} \cdot \frac{1}{M} \sum_{y \in B} |F(y)| \approx 0$ \square

The following example shows that this quasi-proposition may fail for the case of very big K, L such that $\frac{K}{M} \approx \frac{L}{M} \approx 0$.

Example 3 Let Y and T be the same as in the previous examples. Fix a number $K \sim \infty$ such that $K/M \approx 0$ and consider the function $F : Y \rightarrow {}^*\mathbb{R}$ given by the formula:

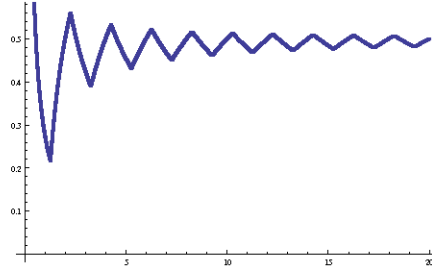


Figure 3: $\Gamma(F)$ for F defined in Eq.10

$$F(k) = \begin{cases} 1, & mK \leq k < (m+1)K, m < R, m \text{ is even} \\ 0, & (mK \leq k < (m+1)K, m < R, m \text{ is odd}) \vee RK \leq m < M \end{cases} \quad (10)$$

The function F is bounded and, thus, S -integrable.

To show that Theorem 6 fails for the set $\{L \sim \infty \mid \frac{L}{M} \approx 0\}$ it is enough to prove that $A_K(F, T, y) \not\approx A_{[\frac{K}{2}]}(F, T, y)$ for all y in some set of the positive measure μ .

Let $D = \bigcup_{m < R} \{y \in Y \mid mK \leq k < mK + \frac{K}{2}\}$. Then, $\mu_L(D) = \frac{1}{2}$. It is easy to see, that $\forall y \in D \forall n \leq \frac{K}{2} F(T^n(y)) = F(y)$, thus, $A_{[\frac{K}{2}]}(F, T, y) = G(y)$.

For every number $n \ll \infty$ consider the set $D_n = \{y \in D \mid |A_K(F, T, y) - F(y)| < \frac{1}{n}\}$. It is enough to prove that $\lim_{n \rightarrow \infty} \mu_L(D_n) = 0$. Since the cardinality of the set $D_n \cap [mK, (m+1)K)$ is the same for all $m < R$, it is enough to calculate the cardinality of $E_n = D_n \cap [0, \frac{K}{2}]$. Since $E_n \subseteq D$, for any $y \in E_n$ one has $A_{[\frac{K}{2}]}(F, T, y) = F(y) = 1$. On the other hand $A_K(F, T, y) = \frac{K-y}{K} = 1 - \frac{y}{K}$. So, $|E_n| = [\frac{K}{n}]$ and $\mu_L(D_n) = (R [\frac{K}{n}]) / (RK + S) \rightarrow 0$. \square

On Fig.3 we see the visual image of the set $\Gamma(F)$ for $M = 100000$, $K = 1000$ and the randomly chosen $y = 870722$.

Theorem 7 *If Y is a hyperfinite set, $|Y| = M$ is a very big number, and $T : Y \rightarrow Y$ is an arbitrary permutation, then for any S -integrable function $F : Y \rightarrow {}^*\mathbb{R}$ there exists an initial segment $I \subseteq {}^*\mathbb{N}_\infty$ such that for almost all $y \in Y$ for all $L, N \in I$ one has $A_L(F, T, y) \approx A_N(F, T, y)$. In the case of transitive permutation T one has*

$$A_N(F, T, y) \approx {}^\circ \text{Av}(F) = \int_Y {}^\circ F d\mu_L \quad (11)$$

for all N such that $\frac{N}{M} \approx 1$.

The statement of this theorem cannot be seen on the on the picture of $\Gamma(F)$, where we see simply some ambiguity around the origin. Theorem 7 states, however, that one can always find a number $N \sim \infty$ such that the visual image of the set of points $\{(n/N, A_n(F, T, y)) \mid n = 1, \dots, N\}$, is a horizontal line (maybe again with some ambiguity around the origin due to numbers $n \ll \infty$) for almost all $y \in Y$. In other words, this means that it is always possible to find a microscope, such that looking through it on the picture of $\Gamma(F)$ around the origin one can always see that the initial part of this graph is a horizontal line and we may use one microscope for almost all $y \in Y$.

On Fig.4 one can see the visual image of the set $\{(n/N, A_n(F, T, y)) \mid n = 1, \dots, N\}$ for the function F of Example 3 where $M = 100000$, $K = 1000$, $N = 0.2K$ for. Fig.4 shows, that $N = 0.4K$ does not satisfy Theorem 7.

The proof of Theorem 7 is not elementary. It is contained in Section 4. It is interesting that this proof uses Ergodic Theorem and Egoroff's Theorem for infinite probability spaces. One can easily show that any $N \ll K$ in Example 3 satisfies Theorem 7.

The following example shows that the statement Theorem of 7 may not be true for all $y \in Y$.

Example 4. Let Y, T be the same as in the previous examples and $F(y) = \frac{y}{M}$. Consider the following function $\psi : [0, 1]^2 \rightarrow \mathbb{R}$ by the formula

$$\psi(a, t) = \begin{cases} t + \frac{a}{2}, & 0 \leq t \leq 1 - a \\ t + \frac{a}{2} - 1 + \frac{1}{a}(1 - t), & 1 - a < t \leq 1 \end{cases}$$

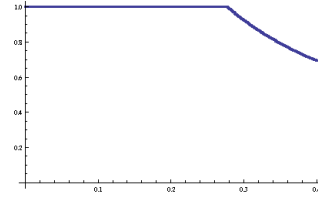
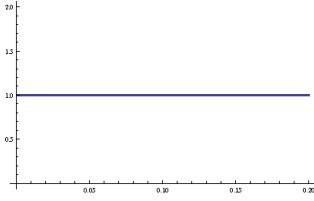


Figure 4: $\Gamma(F)$

Then it is easy to calculate, that

$$A_K(F, T, y) \approx \begin{cases} \psi(0, \frac{y}{M}), & \frac{K}{M} \approx 0, \forall y < M - K \\ \psi(a, \frac{y}{M}) \frac{K}{M} \approx a > 0, & \forall y \in Y \end{cases} \quad (12)$$

Another easy calculation shows that for $K \sim \infty$ such that $\frac{K}{M} \approx 0$, and for $y \in \{K - M, \dots, M - 1\}$ one has $A_K(F, T, y) \approx \infty$. Obviously $\mu(\{K - M, \dots, M - 1\}) = \frac{M - K}{M} \approx 0$.

iii) In this section we introduce definition of approximation of a dynamical system (X, ν, τ) on a compact metric space (X, ρ) with a Borel measure ν and a ν -preserving transformation τ by an internal dynamical system (Y, μ, T) on a hyperfinite set Y , $|Y| = M$ with a uniform probability measure μ and an internal permutation $T : X \rightarrow X$. Since we consider only the measure μ on a hyperfinite space we write (Y, T) for the above hyperfinite dynamical system. For a set $C \subseteq X$ we denote the set $\{x \in X \mid \exists c \in C \rho(x, c) < \varepsilon\}$ by $U_\varepsilon(C)$.

Definition 5 1) Let $\varphi : Y \rightarrow {}^*X$ be an internal injective map such that for every closed set $C \subseteq X$ there exists an initial segment $I \subseteq {}^*\mathbb{N}$ such that $\mu(\varphi^{-1}(U_{\frac{1}{n}}({}^*C))) \approx \nu(C)$ for all $n \in I$. Then the pair (Y, φ) is said to be a hyperfinite approximation (h.a.) of the measure space (X, ν) . In case of $Y \subseteq {}^*X$ and the identical embedding φ we say that Y is a h.a. of (X, ν) . Obviously, any h.a. (Y, φ) is equivalent to the h.a. $\varphi(Y)$.

2) Let $\tau : X \rightarrow X$ be a measure preserving transformation of X and (Y, φ) be a h.a. of (X, ν) . Then we say that an internal permutation $T : X \rightarrow X$ is a h.a. of the transformation τ if for almost all $y \in Y$ one has $\varphi(T(y)) \approx \tau(\varphi(y))$. We say also that the internal triple (Y, T, φ) is a h.a. of the dynamical system (X, ν, τ) .

Proposition 11 A pair (Y, φ) is h.a. approximation of (X, ν) , if the map $st \circ \varphi : Y \rightarrow X$ is a measure preserving map with respect to the measure ν and the Loeb measure μ_L .

Proof By Proposition 4 the condition of Definition 5 (1) is equivalent to the following condition

$$\nu(C) = \lim_{\varepsilon \rightarrow 0} \mu(\varphi^{-1}(U_\varepsilon(C))). \quad (13)$$

It is easy to see that, if C is a compact set, then $\bigcap_{n \in {}^*\mathbb{N}_\infty} U_{\frac{1}{n}}({}^*C) = st^{-1}(C)$. Using this fact and the equality (13) one obtains the equality $\nu(C) = \mu_L(\varphi^{-1}(st^{-1}(C)))$. \square

Let $f \in L_1(\nu)$ and $Y \subseteq {}^*X$ be a h.a. of (X, ν) . Restrict the function $st : {}^*X \rightarrow X$ on the set Y . Then Proposition 11 shows that st is a measure-preserving map and, thus, $f \circ st \in L_1(\mu_L)$. Due to Proposition 9(3) $\varphi \circ st$ has an S -integrable lifting F . We say in this case that F is an S -integrable lifting of the function f . In this case

$$Av(F) = \int_Y F d\mu_L = \int_X f d\nu. \quad (14)$$

Proposition 12 If $f \in C(X)$, then ${}^*f \upharpoonright Y$ is an S -integrable lifting of f .

Theorem 8 Let ν be a non-atomic Borel measure on a compact metric space X such that the measure of every ball is positive. Then

1. for every set $A \subseteq X$ such that $\nu(A) = 1$ there exists a hyperfinite set $Y \subseteq {}^*A$ such that Y is a h.a. of (X, ν) .

2. For every dynamical system (X, ν, τ) and for every h.a. Y of (X, ν) there exists a h.a. (Y, T) . Moreover, one can choose a h.a. T of τ to be a cycle of the length M .

The proof of this theorem is contained in Section 4.

In what follows a Y -cycle of the length M is said to be a *transitive permutation* of Y .

Example 4'. Returning to Example 4 above define the map $\varphi : Y \rightarrow [0, 1]$ by the formula $\varphi(y) = \frac{y}{M}$. Then, obviously, the pair (Y, φ) approximates the probability space $([0, 1], dx)$, where dx is the Lebesgue measure. The permutation T approximates the identical map $id : [0, 1] \rightarrow [0, 1]$. Indeed, for all $y \neq M - 1$ one has $id(\varphi(y)) = j(y) = \frac{y}{M} \approx \frac{y+1}{M} = \varphi(T(y))$. Thus, Y, φ, T and $\tau = id$ satisfy Definition 5. The function F of Example 4 is a lifting of the function $g(x) = x$

Proposition 13 Let (Y, T) be a h.a. of (X, ν, τ) , $f \in L_1(\nu)$, $\tilde{f} = \lim_{n \rightarrow \infty} A_n(f, \tau, x)$ and let \tilde{F} be an S -integrable lifting of \tilde{f} , then there exists an initial segment $I \subseteq {}^*\mathbb{N}_\infty$, such that for almost all $y \in Y$

$$\forall K \in I \quad A_N(F, T, y) \approx \tilde{F}(y) \approx \tilde{f}(\varphi(y)).$$

This proposition follows immediately from Theorem 7.

Corollary 4 Let T be a transitive permutation and let τ be a non-ergodic transformation. Consider a function $f \in L_1(\nu)$ such that the set $B \subseteq X$ of all $x \in X$ satisfying inequality $\lim_{n \rightarrow \infty} A_n(f, \tau, x) \neq Av(f)$ has a positive measure ν . Then there exist infinite M -bounded N, K such that for almost all $y \in st^{-1}(B)$ one has $A_N(F, T, y) \not\approx A_K(F, T, y)$.

Proof. Let $\tilde{f} = \lim_{n \rightarrow \infty} A_n(f, \tau, \cdot)$ and \tilde{F} be the same as in Proposition 13. By this proposition there exists $N \in {}^*\mathbb{N}_\infty$ such that $\frac{N}{M} \approx 0$ and $A_N(F, T, y) \approx \tilde{F}(y)$ μ_L -a.e. Thus, $A_N(F, T, y) \not\approx Av(f)$ for μ_L -almost all $y \in st^{-1}(B)$.

On the other hand, since T is a cycle of length M , by Theorem 6 one has $A_K(F, T, y) \approx Av(F) \approx Av(f)$ for all $y \in Y$ and for all K such that $\frac{K}{M} \approx 1$. Thus, $A_K(F, T, y) \not\approx A_M(F, T, y)$ for μ_L -almost all $y \in st^{-1}(B)$. \square

Proposition shows that if (Y, T) is a h.a. of a dynamical system (X, ν, τ) , τ is a non-ergodic transformation and T is a transitive permutation, then there exists a function $f \in L_1(\nu)$ an internal set $B \subseteq Y$, $\mu(B) \gg 0$ and $K, L \in {}^*\mathbb{N}_\infty$ such that $A_K(F, T, y) \not\approx A_L(F, T, y)$ for all $y \in B$, where F is an S -integrable lifting of f .

Corollary 5 If under the conditions of the previous paragraph, for any $f \in L_1(\mu)$ for almost all $y \in Y$ and for all $N \in {}^*\mathbb{N}_\infty$ one has

$$A_N(F, T, y) \approx \int_X f d\nu, \quad (15)$$

then τ is an ergodic transformation

We do not know, whether the sufficient condition of the ergodicity of τ formulated in Corollary 5, is also a necessary condition. By Proposition 13 the approximate equality (15) holds for all N in some initial segment $I \subseteq {}^*\mathbb{N}_\infty$ for almost all $y \in Y$ for an ergodic transformation τ or for any transformation τ and transitive permutation T , if $\frac{N}{M} \approx n \in \mathbb{N}$ or, if $\frac{N}{M} \sim \infty$ (see Theorem 10).

This and even stronger is necessary for uniquely ergodic transformations. Recall that a continuous transformation $\tau : X \rightarrow X$ is said to be *uniquely ergodic* if there exists only one τ -invariant Borel measure on X ⁵.

Theorem 9 If τ is a uniquely ergodic transformation of a compact metric space X , $Y \subseteq {}^*X$ is a hyperfinite dense subset of *X , and $T : Y \rightarrow Y$ is an internal permutation such that $\forall y \in Y \quad st(T(y)) = \tau(st(y))$, then for every $y \in Y$ such that the τ -orbit of $st(y)$ is dense in X , for every $N \in {}^*\mathbb{N}_\infty$ and for every $f \in C(X)$ one has

$$A_N(*f \upharpoonright Y, T, y) \approx \int_X f d\nu, \quad (16)$$

where ν is the τ -invariant measure.

⁵Krylov-Bogoljubov theorem claims the existence of at least one τ -invariant measure.



Figure 5: $\Gamma(F)$ for transformation $\tau(x) = x + 2/3 \mod 1$



Figure 6: $\Gamma(f)$ for transformation $\tau(x) = x + 1/\sqrt{2} \mod 1$

The proof of this theorem is contained in Section 4.

Example 5. In this example we consider a hyperfinite set $Y = \{0, 1, \dots, M-1\}$ and a permutation $T : Y \rightarrow Y$ given by the formula $T(y) = y + P \mod M$. We choose P and M to be relatively prime, so that T is a cycle of length M . The approximation (Y, φ) of the probability space $([0, 1], dx)$ used in Example 4' can be used as well for the probability space $([0, 1], dx)$, where $[0, 1]$ is provided by the topology of the circle. For any $t \in \mathbb{R}$ the measure preserving transformation $\tau : [0, 1] \rightarrow [0, 1]$, such that $\tau(y) = y + t \mod 1$ is called the t -shift of a circle. This transformation is continuous on $[0, 1]$. It is easy to see that if $\frac{P}{M} \approx t$, then the triple (Y, φ, T) approximate the shift τ . We present the visual images of $\Gamma(F)$ for $F = f \upharpoonright \varphi(Y)$, where

$$f(x) = \begin{cases} \frac{10x}{9}, & \text{if } 0 \leq x < 0.9 \\ 10(1-x), & \text{if } 0.9 \leq x < 1 \end{cases}$$

We choose the function f that is close to the function $g(x) = x$, considered in Example 4'. However, f is continuous on the circle $[0, 1]$, while g is discontinuous at $x = 0$. We consider two cases.

a). On Fig.5 the visual image of $\Gamma(F)$ on the interval $[0, 1]$ and at the neighborhood of 0 are presented for the case of $M = 33334$, $P = 22225$ and the randomly chosen $y = 16667$.

b). On Fig.6 we see the visual image of $\Gamma(f)$ on the interval $[0, 1]$ and at the neighborhood of 0 are presented for the case of $M = 25001$, $P = 17677$ and the randomly chosen $y = 6119$. In this case (Y, φ, T) approximates $1/\sqrt{2}$ -shift. In accordance with Theorem 9 this visual image is a horizontal line.

To explain the difference in the graphs in the cases a) and b) notice that in the case a) $\frac{P}{M} \approx \frac{2}{3}$. Actually, $|\frac{P}{M} - \frac{2}{3}| \leq 0.00046$ that is enough for our problem to consider these numbers to be infinitesimally close (see the discussion in the Example 1). In this case $x = \frac{y}{M} = \frac{16667}{33334} \approx 0.5$ and $\tau(y) = y + \frac{2}{3} \mod 1$. It is easy to see that for $\frac{1}{3} < y < \frac{2}{3}$ one has $\hat{\varphi}(y) = \frac{1}{3}[\varphi(y - 1/3) + \varphi(y) + \varphi(y + 1/3)]$ (for any integrable function φ). So, in our case $\hat{\varphi}(0.5) \approx 0.56$. We see that the value of the function on Fig.5 at the neighborhood of 0 is close to 0.56 that agrees with Proposition 13. Since T is a cycle of length M the value of the function on Fig.5 at the neighborhood of 1 is close to $Av(f) = Av(\varphi \upharpoonright j(X)) \approx \int_0^1 \varphi d\mu$ by Definition 5. In our case $\int_0^1 \varphi dx = 0.5$. The visual image of $\Gamma(f)$ is a continuous function on $[0, 1]$ in accordance with Theorem 6.

In case b) $\frac{P}{M} \approx \frac{1}{\sqrt{2}}$ ($|\frac{P}{M} - \frac{1}{\sqrt{2}}| \leq 0.00006$). So, in this case T approximates irrational shift of the circle $[0, 1]$. It is well-known that irrational shifts of the circle are uniquely ergodic transformations. Since φ is a continuous function,

$A_K(f, T, x) \approx 0.5$ for all $K \sim \infty$ by Theorem 9. Thus, the visual image of $\Gamma(f)$ is the horizontal line $y = 0.5$, that is perfectly reflected on both pictures of Fig.6.

The consideration of these two examples arises the following question. Suppose that we have a ratio $\frac{P}{M}$ of two relatively prime numbers. In what case this ratio can be considered as "practically" rational number and in what case one should consider it as "practically" irrational number. Using the informal language of the Introduction one can say that that $\frac{P}{M}$ is "practically" rational, if there exist two natural numbers $m, n \ll \infty$ such that $\frac{P}{Q} \approx \frac{m}{n}$, and $\frac{P}{M}$ is "practically" irrational otherwise. Certainly the exact answer strongly depends on a problem, in which we need to answer this question. However, the statements of this section provide us with some qualitative understanding of the correlation between the behaviors of very big discrete systems and their continuous approximations, (or vice versa, continuous systems and their discrete approximations).

Example 6. (Approximations of Bernoulli shifts). Let $\Sigma_m = \{0, 1, \dots, m-1\}$. Consider the compact space $X = \Sigma_m^{\mathbb{Z}}$ with the Tychonoff topology. Let a be a function, such that $\text{dom}(a) \subset \mathbb{Z}$ is finite, and $\text{range}(a) \subseteq \Sigma_m$. Let $S_a = \{f \in X \mid f \upharpoonright \text{dom}(a) = a\}$. Then the family of all such S_a form a base of neighborhoods of the compact space X . For $g \in {}^*X$ set $f = g \upharpoonright \mathbb{Z}$, then $f \in X$ and it is easy to see that $f = \text{st}(g)$.

The continuous transformation $\tau : X \rightarrow X$ defined by the formula $\tau(f)(n) = f(n+1)$ where $f \in X$ and $n \in \mathbb{Z}$ is an invertible Bernoulli shift. Every probability distribution $\{p_0, \dots, p_{m-1}\}$ ($p_i > 0$, $\sum_{i=0}^{m-1} p_i = 1$) on Σ_m defines a Borel measure on X that is obviously invariant with respect to τ . It is well-known that τ is ergodic for each of these measures. So, the transformation τ is not uniquely ergodic. Here we restrict ourselves only to the case of the uniform distribution on Σ_m , i.e. to the case of $p_0 = \dots = p_{m-1} = \frac{1}{m}$. The corresponding Borel measure on X is denoted by ν . Obviously, $\nu(S_a) = m^{-|\text{dom}(a)|}$.

We construct here two hyperfinite approximations of the dynamical system (X, ν, τ) . First we consider the straightforward approximation by a hyperfinite shift. Fix $N \in {}^*\mathbb{N}_\infty$ and set $Y = \Sigma_m^{\{-N, \dots, N\}}$. Then $M = |Y| = m^{2N+1}$. Define $\lambda : Y \rightarrow {}^*X$ as follows. For $y \in \Sigma_m^{\{-N, \dots, N\}}$ set

$$\lambda(y)(n) = \begin{cases} y(n), & |n| \leq N \\ 0, & |n| > N \end{cases} \quad (17)$$

Then $\text{st} \circ \lambda(y) = y \upharpoonright \mathbb{Z}$. Thus, for every standard neighborhood S_a defined above one has $(\text{st} \circ \lambda)^{-1}({}^*S_a) = \{y \in Y \mid y \upharpoonright \text{dom}(a) = a\}$. So, $\mu_L((\text{st} \circ \lambda)^{-1}({}^*S_a)) = \nu(S_a) = m^{-|\text{dom}(a)|}$. This proves that (Y, λ) is a h.a. of (X, ν) .

Certainly, an arbitrary internal map from Y to $\Sigma_m^{\mathbb{Z} \setminus \{-N, \dots, N\}}$ can be used to define the values $\lambda(y)(n)$ for $|n| > N$ and $y \in Y$ in the definition of λ (17)

In what follows we use notations $y_1 \approx y_2$ and $\text{st}(y)$ for $\lambda(y_1) \approx \lambda(y_2)$ and $\text{st}(\lambda(y))$ respectively.

Define the map $S : Y \rightarrow Y$ by the formula $S(y)(n) = y(n+1 \pmod{2N+1})$ for any $y \in Y$ and $n \in \{-N, \dots, N\}$. Then obviously $\tau(\text{st}(y)) = \text{st}(S(y))$ for all $y \in Y$. So, (Y, λ, S) is a h.a. of the dynamical system (X, ν, τ) .

Since every point $y \in Y$ is $(2N+1)$ -periodic with respect to S the permutation S is not transitive. Though the existence of a transitive h.a. of τ is proved in Theorem 8, it is not easy to construct such approximation explicitly.

To do this we reproduce here the construction of de Bruijn sequences.

Definition 6 An (m, n) -de Bruijn sequence on the alphabet Σ_m is a sequence $s = (s_0, s_1, \dots, s_{L-1})$ of $L = m^n$ elements $s_i \in \Sigma_m$ such that all consecutive subsequences $(s_i, s_{i+1}, \dots, s_{i+n-1})$ of length n are distinct.

Here and below in this example the symbols \oplus and \ominus denote $+$ and $-$ modulo L , so that the sequence s is considered as a sequence of symbols from Σ_n placed on a circle.

It was proved [4, 5] that there exist $(m!)^{m^{n-1}} \cdot m^{-n}$ (m, n) -de Bruijn sequences. See also [6] for a simple algorithm for de Bruijn sequences and more recent references.

To construct a transitive h.a. $T : Y \rightarrow Y$ of τ fix arbitrary $(m, 2N+1)$ de Bruijn sequence $s = (s_0, s_1, \dots, s_{M-1})$ here $L = M$. Let $y = (y_{-N}, \dots, y_N) \in Y$. Then there exists the unique consecutive subsequence $\sigma(y) = (s_i, s_{i+1}, \dots, s_{i+2N})$ such that $y_j = s_{j+i \oplus N}$. Set $P(\sigma(y)) = (s_{i+1}, \dots, s_{i+2N+1})$ and $T(y) = \sigma^{-1}(P(\sigma(y)))$. Notice that if $i < M - N$, then for all $j \leq N$ one has $s_{j+i \oplus N} = s_{j+i+N}$. So, $T(y)_j = y_{j+1}$ for all $j < N$ and, thus, for all standard j . This last equality implies that $\text{st}(T(y)) = \tau(\text{st}(y))$ for all $y \in Y$ such that the first entry of the sequence $\sigma(y)$ is the i -s term of the initial de Bruijn sequence for $i \leq L - 2N - 1$. So, $\mu_L(\{y \mid \text{st}(T(y)) = \tau(\text{st}(y))\}) \geq \frac{M-N}{M} \approx 1$. This proves that T is a h.a. of τ . We call T a de Bruijn approximation of τ .

It is interesting to study the behavior of ergodic means of described approximations. This problem will be discussed in another paper. We confine ourselves with two simple remarks.

1. If $\sigma(y) = \langle s_i, \dots, s_{i+2N} \rangle$ and $i < M - N$, then $A_n(F, T, y) = A_n(F, S, y)$ for all $n < N$.
2. Let $S_0 = \{f \in X \mid f(0) = 1\}$, so that $\nu(S_0) = \frac{1}{2}$ and let χ_0 be a characteristic function of S_0 . For $y \in Y$ let $f = \text{st}(y)$. Set $A(y) = f^{-1}(\{1\}) \cap \mathbb{N}$. Recall that the density of $A(y)$ is given by the formula

$$d(A(y)) = \lim_{m \rightarrow \infty} \frac{|A(y) \cap \{0, \dots, m-1\}|}{m}.$$

It is easy to see that for $m < N$ one has

$$A_m(*\chi_0, T, y) = \frac{|A(y) \cap \{0, \dots, m-1\}|}{m}.$$

So, for all $y \in Y$ such that the density $d(A(y))$ exists one has $\exists K \in {}^*\mathbb{N}_\infty \forall m \in {}^*\mathbb{N}_\infty \quad m \leq K \implies A_m(\chi_0, T, y) \approx d(A(y))$. Due to Proposition 13 there exist $K \in {}^*\mathbb{N}_\infty$ such that for μ_L -almost all $y \in Y$ one has $A_m(*\chi_0, T, y) \approx \frac{1}{2}$.

iv) (*Formulation of results in the framework of the standard mathematics.*) While in nonstandard analysis we use the notion of an infinite number (hyperfinite set) as a formalization of the notion of a very big number (finite set), in classical mathematics we use the sequences of numbers (finite sets) diverging to infinity to formalize these notions. For example, in previous sections we considered a hyperfinite set Y and its internal permutation $T : Y \rightarrow Y$. If we want to treat the same problems in the framework of standard mathematics, we have to consider a sequence (Y_n, T_n) of finite sets Y_n whose cardinalities tend to infinity and their permutations T_n . Similarly, internal functions $F : Y \rightarrow {}^*\mathbb{R}$ correspond to sequences $F_n : Y_n \rightarrow \mathbb{R}$ in standard mathematics.

First, we discuss what property of such sequences correspond to the property of an internal function F to be S -integrable. The following proposition gives a reasonable answer to this question.

Proposition 14 *Let Y_n be a standard sequence of finite sets, such that $|Y_n| = M_n \rightarrow \infty$ as $n \rightarrow \infty$. Then for an arbitrary sequence $F_n : Y_n \rightarrow \mathbb{R}$ the following statements are equivalent:*

1. For every $K \in {}^*\mathbb{N}_\infty$ the function $*F_K$ is S -integrable.
- 2.

$$\lim_{n, k \rightarrow \infty} \frac{1}{M_n} \sum_{\{x \in Y_n \mid |F_n(x)| > k\}} |F_n(x)| = 0 \quad (18)$$

The proof can be obtained easily from the definition of S -integrable functions (5) using arguments similar to those that were used in the proof of Theorem 3

A sequence F_n that satisfies the statement (2) of Proposition 14 is said to be *uniformly integrable*.

Proposition 14 leads to establishing the standard version of Theorem 6.

Proposition 15 *In conditions of Proposition 14 let $T_n : Y_n \rightarrow Y_n$ be a sequence of transitive permutations and $F_n : Y_n \rightarrow \mathbb{R}$ be a uniformly integrable sequence. Consider two sequences of natural numbers K_n and L_n such that $\frac{K_n}{M_n}$ is bounded, $\liminf \frac{K_n}{M_n} > 0$ and $\lim_{n \rightarrow \infty} \frac{K_n}{L_n} = 1$. Then the following two statements are true.*

1. For any $\varepsilon > 0$ one has

$$\lim_{n \rightarrow \infty} \frac{1}{M_n} \cdot |\{y \in Y_n \mid |A_{K_n}(F_n, T_n, y) - A_{L_n}(F_n, T_n, y)| \geq \varepsilon\}| = 0 \quad (19)$$

2. If T_n is a sequence of transitive permutations or F_n is a sequence of bounded functions, then

$$\lim_{n \rightarrow \infty} \max_{y \in Y_n} |A_{K_n}(F_n, T_n, y) - A_{L_n}(F_n, T_n, y)| = 0 \quad (20)$$

It is not too difficult to rewrite the proof of Theorem 6 in (standard) terms of Proposition 15. It is also easy to deduce Proposition 15 from Theorem 6 using arguments close to those of the proof of Theorem 3.

The rigorous interpretation of Theorem 7 in the framework of standard mathematics is much more difficult. To formulate the corresponding rigorous theorem, we need to remind the notion of an ultrafilter and of the limit of a sequence over a non-principle ultrafilter.

Recall that a subset $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is said to be a *non-principle ultrafilter*, if $\mathcal{F} = \{A \subseteq \mathbb{N} \mid m(A) = 1\}$ for some finitely additive measure m on $\mathcal{P}(\mathbb{N})$ that assumes only two values 0 and 1 such that $m(B) = 0$ for any finite set $B \subseteq \mathbb{N}$ and $m(\mathbb{N}) = 1$. For a sequence $\alpha : \mathbb{N} \rightarrow \mathbb{R}$ we say that $\lim_{\mathcal{F}} \alpha = L$, if for any $\varepsilon > 0$ the set $\{n \in \mathbb{N} \mid |\alpha(n) - L| < \varepsilon\} \in \mathcal{F}$.

It is known that any bounded sequence has a limit over any non-principle ultrafilter. For two sequences $\alpha, \beta : \mathbb{N} \rightarrow \mathbb{R}$ we say that $\alpha \leq_{\mathcal{F}} \beta$ iff $\{n \in \mathbb{N} \mid \alpha(n) \leq \beta(n)\} \in \mathcal{F}$.

Proposition 16 *Let $T_n : Y_n \rightarrow Y_n$ be a sequence of arbitrary permutations. Then for every non-principle ultrafilter $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ there exists a sequence $K_n \rightarrow \infty$ as $n \rightarrow \infty$ and a sequence $A_n \subseteq Y_n$ such that $\lim_{\mathcal{F}} \frac{|A_n|}{M_n} = 0$ and for any $L_n \rightarrow \infty$ as $n \rightarrow \infty$, if $\langle L_n \rangle \leq_{\mathcal{F}} \langle K_n \rangle$, then $\lim_{\mathcal{F}} (A_{K_n}(f_n, T_n, x_n) - A_{L_n}(f_n, T_n, x_n)) = 0$.*

This proposition doesn't have such intuitively clear sense as Theorem 7. One hardly can find a proof of it, that doesn't use the ideas of nonstandard analysis.

To formulate the standard version of Definition 5 introduce the following notation. Let $Z \subseteq X$ be a finite subset of X and $\delta_Z = \frac{1}{|Z|} \sum_{z \in Z} \delta_z$, where δ_z is a Dirac measure at a point $z \in Z$, i.e. δ_z is a Borel probability measure such that for any Borel set $A \subseteq X$ one has $\delta_z(A) = 1 \iff z \in A$. Obviously, for any function $f \in C(X)$ one has

$$\int_X f d\delta_Z = \frac{1}{|Z|} \sum_{z \in Z} f(z).$$

Definition 7 *In conditions of Definition 5 let $\{Y_n \mid n \in \mathbb{N}\}$ be a sequence of finite subsets of X . We say that the sequence Y_n approximates the measure space (X, ν) if the sequence of measures δ_{Y_n} converges to the measure ν in the $*$ -weak topology on the space $\mathcal{M}(X)$ of all Borel measures on X .*

The following proposition follows easily from well known theorems of functional analysis.

Proposition 17 *In conditions of Definition 7 suppose that every open ball in X has the positive measure ν and every set of the positive measure ν is infinite. Then for every set $A \subseteq X$ with $\nu(A) = 1$ there exists a sequence Y_n of finite subsets of X approximating the measure space (X, ν) such that $\forall n \in \mathbb{N} Y_n \subseteq A$. \square*

Definition 7 can be considered as a standard sequence version of Definition 5 (1). This statement is based on the following

Proposition 18 *A sequence $Y_n \subseteq X$ approximates a measure space (X, ν) in the sense of Definition 7 if and only if for any $N \in {}^*\mathbb{N}_{\infty}$ the set Y_N is a hyperfinite approximation of the measure space (X, ν) .*

Proof. \implies Let Y_n approximates (X, ν) and $N \in {}^*\mathbb{N}_{\infty}$. Then for any $f \in C(X)$ one has

$$\int f(st(x)) d\mu_L \circ \left(\frac{1}{|Y_N|} \sum_{y \in Y_N} {}^*f(y) \right) = \int_X f d\nu \quad (21)$$

The first equality is due to *f is a lifting of $f \circ st$. The second follows from Definition 7 and from the nonstandard analysis definition of the limit of a sequence. Now $st : Y_N \rightarrow X$ defines a measure ν' on X that is the image of the Loeb measure of Y . Due to (21) and the Riesz representation theorem $\nu' = \nu$.

\Leftarrow Assume that $st \upharpoonright Y_N : Y_N \rightarrow X$ is a measure preserving transformation for every $N \in {}^*\mathbb{N}_{\infty}$. It is easy to see that for every function $f \in C(X)$ the internal function ${}^*f \upharpoonright Y_N$ is a lifting of f . So,

$$\circ \left(\frac{1}{|Y_N|} \sum_{y \in Y_N} {}^*f(y) \right) = \int_{Y_N} \circ({}^*f) d\mu_L = \int_{Y_N} f \circ st d\mu_L = \int_X f d\nu.$$

Thus, the equality (21) holds for every $N \in {}^*\mathbb{N}_\infty$ and by the nonstandard analysis definition of a limit one has

$$\lim_{n \rightarrow \infty} \int_X f d\delta_{Y_n} = \int_X f d\nu \quad \square$$

The last two propositions imply Theorem 8(1).

We use the same approach as above to formulate a sequence version of the notion of a hyperfinite approximation of a dynamical system.

Definition 8 Let (X, ρ) be a compact metric space, ν be a Borel measure on X , $\tau : X \rightarrow X$ be a measure preserving transformation of X , $\{Y_n \subseteq X \mid n \in \mathbb{N}\}$ be a sequence of finite approximation of the measure space (X, ν) in the sense of Definition 7 and $T_n : Y_n \rightarrow Y_n$ be a sequence of permutations of Y_n . We say that a sequence T_n is an approximating sequence of the transformation τ if for every $N \in {}^*\mathbb{N}_\infty$ the internal permutation $T_N : Y_N \rightarrow Y_N$ is a h.a. of τ in the sense of Definition 5. In this case we say that the sequence of finite dynamical systems (Y_n, μ_n, T_n) approximates the dynamical system (X, ν, τ) . Here μ_n is a uniform probability measure on Y_n .

The reformulation of this definition in full generality in standard mathematical terms is practically unreadable. However, it is easy to reformulate it for the case of an almost everywhere continuous transformation τ . This case covers a lot of important applications.

Denote the set of all points of continuity of the map $\tau : X \rightarrow X$ by D_τ .

Lemma 1 Suppose that $\nu(D_\tau) = 1$ and let $Y \subseteq X$ be a h.a. of the measure space (X, ν) . Then a permutation $T : Y \rightarrow Y$ is a h.a. of the transformation τ if and only if for every positive $\varepsilon \in \mathbb{R}$ one has

$$\frac{1}{|Y|} (|\{y \in Y \mid \rho(T(y), {}^*\tau(y)) > \varepsilon\}|) \approx 0. \quad (22)$$

Proof (\implies) Let $A = \{y \in Y \mid {}^\circ y \in D_\tau\} = \text{st}^{-1}(D_\tau)$, $B = \{y \in Y \mid T(y) \approx \tau({}^\circ y)\}$. Then, $\mu_L(A) = 1$ since Y is a h.a. of the measure space (X, ν) and $\nu(D_\tau) = 1$. Since T is a h.a. of τ , one has $\mu_L(B) = 1$. Thus, $\mu_L(A \cap B) = 1$. Since τ is continuous on Δ_τ one has

$$\forall x \in X \quad {}^\circ x \in D_\tau \implies {}^*\tau(x) \approx \tau({}^\circ x). \quad (23)$$

So, $\forall y \in A \cap B \quad {}^*\tau(y) \approx \tau({}^\circ y)$ and thus, $\forall y \in A \cap B \quad {}^*\tau(y) \approx T(y)$. So, for every positive $\varepsilon \in \mathbb{R}$ one has $\{y \in Y \mid \rho(T(y), {}^*\tau(y)) > \varepsilon\} \subseteq Y \setminus (A \cap B)$. This proves (22).

(\impliedby) Suppose that (22) holds for every positive $\varepsilon \in \mathbb{R}$. Then obviously $\mu_L(\{y \in Y \mid T(y) \approx {}^*\tau(y)\}) = 1$. On the other hand, by (23) one has $\mu_L(\{y \in Y \mid {}^*\tau(y) \approx \tau({}^\circ y)\}) = 1$. Thus, $\mu_L(\{y \in Y \mid T(y) \approx \tau({}^\circ y)\}) = 1$, i.e. T is a h.a. of τ \square

Lemma 1 implies immediately the following

Proposition 19 (Standard version of Definition 8) In conditions of Definition 8 and Lemma 1 the sequence of permutations $T_n : Y_n \rightarrow Y_n$ is an approximating sequence of the transformation τ if and only if for every positive $\varepsilon \in \mathbb{R}$ one has

$$\lim_{n \rightarrow \infty} \frac{1}{|Y_n|} (|\{y \in Y_n \mid \rho(T_n(y), \tau(y)) > \varepsilon\}|) = 0. \quad \square \quad (24)$$

Now we are able to prove the sequence version of Theorem 8 (2).

Theorem 10 Let (X, ρ) be a compact metric space and ν be a Borel measure on X such that the measure space (X, ν) satisfies the conditions of Theorem 8. Then for every measure preserving transformation $\tau : X \rightarrow X$ with $\nu(D_\tau) = 1$ there exist a sequence of finite sets $Y_n \subseteq X$ and a sequence of permutations $T_n : Y_n \rightarrow Y_n$ such that the sequence of finite dynamical systems (Y_n, T_n) approximates the dynamical system (X, ν, τ) in the sense of Definition 8. Moreover, one can choose transitive permutations T_n .

Proof. Let $Y_n \subseteq X$ be a sequence that approximates the measure space (X, ν) in the sense of Definition 7. Such sequence exists by Proposition 17. Then by Proposition 18 for any $N \in {}^*\mathbb{N}_\infty$ the set Y_N is a h.a. of the measure space (X, ν) in the sense of Definition 5. By Theorem 8 there exists a (transitive) permutation $T_N : Y_N \rightarrow Y_N$ that is a h.a. of the transformation τ . By Lemma 1, since $\nu(D_\tau) = 1$, this means that (Y_N, T_N) satisfies (22) for every standard positive ε . In this proof the letter T maybe with lower indexes always denotes a (transitive) permutation.

For every numbers $n, m \in \mathbb{N}$ define the set

$$A_{n,m} = \left\{ k \in \mathbb{N} \mid \exists T : Y_k \rightarrow Y_k \left(\frac{1}{|Y_k|} \cdot \left| \{ y \in Y_k \mid \rho(T(y), \tau(y)) > \frac{1}{n} \} \right| < \frac{1}{m} \right) \right\}.$$

Since $\forall N \in {}^*\mathbb{N}_\infty \ N \in {}^*A_{n,m}$, there exists a standard function $N(n, m)$ such that $\forall k > N(n, m) \ k \in A_{n,m}$. By the definition of sets $A_{m,n}$, there exists a standard function $T(k, n, m) : Y_k \rightarrow Y_k$ with the domain $\{(n, m, k) \in \mathbb{N}^3 \mid k > N(n, m)\}$ such that

$$\frac{1}{|Y_k|} \cdot \left| \left\{ y \in Y_k \mid \rho(T_k(y), \tau(y)) > \frac{1}{n} \right\} \right| < \frac{1}{m}.$$

Now it is easy to see that if $r = N(n, n) + n$, then the sequence (Y_r, T_r) satisfies the conditions of Proposition 19 \square
This proof is based on NSA. The purely standard proof of this theorem seems to be much more difficult.

4 Proofs of Theorems 7, 8(2) and 9.

i) (*Proof of Theorem 7*). We notice first, that the following proposition follows immediately from Theorem 5 applied to the external dynamical system (Y, μ_L, T) and Theorem 3.

Proposition 20 *In conditions of Theorem 6 for any $y \in Y$ there exists an initial segment $I \subseteq {}^*\mathbb{N}_\infty$ such that*

$$\forall L, K \in I \ (A_K(F, T, y) \approx A_L(F, T, y) \approx \lim_{n \rightarrow \infty} A_n({}^\circ F, T, y)).$$

Proposition 20 is a weaker version of Theorem 7, since it differs of this theorem only in the order of quantifiers "for all $y \in T$ and "there exists an initial segment $I \subseteq {}^*\mathbb{N}_\infty$ ".

To prove Theorem 7 first it is necessary to prove

Theorem 11 *Let $f_n : Y \rightarrow \mathbb{R}$, $n \in \mathbb{N}$ be a sequence of μ_L measurable functions on Y , and $F_n : Y \rightarrow {}^*\mathbb{R}$, $n \in {}^*\mathbb{N}$ be an internal sequence such that $\forall n \in \mathbb{N}$, F_n is a lifting of f_n . Then f_n converges to a measurable function f μ_L -almost everywhere if and only if there exists $K \in {}^*\mathbb{N}_\infty$ such that μ_L -almost everywhere $\forall N \in {}^*\mathbb{N}_\infty$, $N < K \implies F_N(x) \approx F(x)$, where F is a lifting of f .*

Proof of Theorem 11.

(\implies) Let f_n converges to f a.e. By Egoroff's theorem

$$\forall k \in \mathbb{N} \exists B_k \subseteq Y \ (\mu_L(B_k) \geq 1 - \frac{1}{k} \wedge f_n(x) = {}^\circ F_n(x) \text{ converges uniformly on } B_k).$$

WLOG we may assume that B_k is internal, $\frac{|B_k|}{|Y|} \geq 1 - \frac{1}{k}$, and $\forall n, k \in \mathbb{N} \forall x \in B_k \ F_n(x) \approx f_n(x)$ and $F(x) \approx f(x)$. Then

$$\exists {}^{\text{st}}\varphi_k : \mathbb{N} \rightarrow \mathbb{N} \forall {}^{\text{st}}r \forall {}^{\text{st}}m > \varphi_k(r) \max_{x \in B_k} |F_m(x) - F(x)| < \frac{1}{r}.$$

Consider the internal set

$$C_r^k = \{N \in {}^*\mathbb{N} \mid \forall m (N > m > \varphi_k(r) \implies \forall x \in B_k \ |F_m(x) - F(x)| < \frac{1}{r})\}$$

The previous statement shows that C_r^k contains all standard numbers that are greater than $\varphi_k(r)$. Thus, there exists infinite $N_r^k \in C_r^k$. By Theorem 2 (5) $\exists K \in {}^*\mathbb{N}_\infty \ \forall {}^{\text{st}}k, r \ K < N_r^k$. Obviously, this K satisfies Theorem 11

(\Leftarrow) Let $B = \{x \in Y \mid \forall N \in {}^*\mathbb{N}_\infty \ N \leq K \implies F_N(x) \approx F(x)\}$, $A_n = \{x \in Y \mid f_n(x) \approx F_n(x)\}$, $n \in \mathbb{N}$, $A = \{x \in Y \mid F(x) \approx f(x)\}$, $C = B \cap A \cap \bigcap_{n \in \mathbb{N}} A_n$.

By conditions of the theorem $\mu_L(C) = 1$. Fix an arbitrary $x \in C$, and an arbitrary $r \in \mathbb{N}$. The internal set $D = \{n \in {}^*\mathbb{N} \mid |F_n(x) - F(x)| \leq \frac{1}{r}\}$ contains all infinite numbers that are less or equal to K . So $\exists n_0 \in \mathbb{N} \forall n > n_0 \ |F_n(x) - F(x)| \leq \frac{1}{r}$. Since $F_n(x) \approx f_n(x)$, the same holds for $f_n(x)$ and ${}^\circ F(x)$. Thus, f_n converges to $f = {}^\circ F$ a.e. \square .

Now we are able to complete the proof of Theorem 7. In conditions of Theorem 7 let $f = {}^\circ F$ and $f_n(x) = A_n(f, T, x)$, $n \in \mathbb{N}$ and the internal sequence $F_n(x) = A_n(F, T, x)$, $n \in {}^*\mathbb{N}$. Then $f \in L_1(\mu_L)$ and F_n is an S -integrable lifting of f_n for all $n \in \mathbb{N}$. By Theorem 5 f_n converges to an integrable function \hat{f} a.e. Let \hat{F} be an S -integrable lifting of \hat{f} . Then by Theorem 11 there exists $K \in {}^*\mathbb{N}_\infty$ such that μ_L -almost surely $\forall N \in {}^*\mathbb{N}_\infty$ $N < K \implies F_N(X) \approx \hat{F}(x)$ \square .

ii) (*Proof of Theorem 8 (2).*) It is well-known (see e.g. [3, 2]) the measure space (X, ν) is a Lebesgue space, i.e. it is isomorphic modulo measure 0 to the measure space $([0, 1], dx)$, where dx is the standard Lebesgue measure. This means that there exist a set $B \subseteq X$ a set $C \subseteq [0, 1]$ and a bijective map $\psi : B \rightarrow C$ such that $dx(C) = \nu(B) = 1$ and the maps ψ, ψ^{-1} are measure preserving.

Lemma 2 *In conditions of the previous paragraph let Y be a h.a. of (X, ν) . Then for every set $D \subseteq [0, 1]$ with $dx(D) = 1$ there exists a bijective lifting $G : Y \rightarrow {}^*[0, 1]$ of the map ψ such that*

1. $Z = G(Y) \subseteq {}^*D$;
2. Z is a h.a. of $([0, 1], dx)$.
3. $G^{-1} : Z \rightarrow {}^*X$ is a lifting of ψ^{-1} .

Proof. Let $F : Y \rightarrow {}^*[0, 1]$ be a lifting of ψ . Let $\sigma = \frac{1}{2} \min\{\rho(u, v) \mid u, v \in F(Y), u \neq v\}$. Then $0 < \sigma \approx 0$ and $\forall u \in F(Y)$ $B_\sigma(u) \cap F(Y) = \{u\}$. Since $\nu(B_\sigma(u)) > 0$ and $dx(D) = 1$ the set $B_\sigma(u) \cap {}^*D$ contains infinitely many points, and thus, there exists an internal set $E_u \subseteq B_\sigma(u) \cap {}^*D$ such that $|E_u| = |F^{-1}(u)|$. Establishing bijection between $F^{-1}(u)$ and E_u for every $u \in F(Y)$, we obtain the bijection $G : Y \rightarrow Z \subseteq {}^*D$ that is a lifting of ψ . Notice that since G and G^{-1} are bijections they are measure preserving maps between measure spaces (Y, μ_L^Y) and (Z, μ_L^Z) .

To prove the second property of the set Z , one needs to show that $st_{[0,1]} \upharpoonright Z : Z \rightarrow [0, 1]$ is a measure preserving map, i.e. that for every measurable set $A \subseteq [0, 1]$ one has

$$\mu_L^Z(st_{[0,1]}^{-1}(A) \cap Z) = dx(A) \quad (25)$$

One has

$$\mu_L^Z(st_{[0,1]}^{-1}(A) \cap Z) = \mu_L^Y(\{y \in Y \mid G(y) \in st_{[0,1]}^{-1}(A)\}) = \mu_L^Y(\{y \in Y \mid {}^\circ G(y) \in A\}). \quad (26)$$

Since G is a lifting of ψ on has ${}^\circ G(y) = \psi(st_X(y))$ for μ_L^Y -almost all y . Thus,

$$\mu_L^Y(\{y \in Y \mid {}^\circ G(y) \in A\}) = \mu_L^Y(\{y \in Y \mid \psi(st_X(y)) \in A\}) = \nu(\psi^{-1}(A)) = dx(A), \quad (27)$$

since $st_X \upharpoonright Y : Y \rightarrow X$ and $\psi : B \rightarrow C \subseteq [0, 1]$ are measure preserving maps. The equality (25) follows from the equalities (26) and (27).

To prove the third property of the set Z it is enough to show that $st_Y(G^{-1}(z)) = \psi^{-1}(st_{[0,1]}(z))$ for μ_L^Z -almost all $z \in Z$. Since ψ is a bijection, the last equality is equivalent to the equality $\psi(st_Y(G^{-1}(z))) = st_{[0,1]}(z)$, which follows from the following sequence of equalities that hold for μ_L^Z -almost all $z \in Z$:

$$\psi(st_Y(G^{-1}(z))) = st_{[0,1]}(G(G^{-1}(z))) = st_{[0,1]}(z). \square$$

The proof of Theorem 8(2) is divided in six parts I – VI.

I. Here we prove the existence of a h.a. (Y, μ_L, T) of the dynamical system $([0, 1], dx, \tau)$. Let Y be an arbitrary h.a. of the measure space $([0, 1], dx)$. Let $F : Y \rightarrow {}^*[0, 1]$ be a lifting of τ . First we prove the following statement.

(A) *For every standard $\delta > 0$ there exists a permutation $T_\delta : Y \rightarrow Y$ such that*

$$\frac{|\{y \in Y \mid |F(y) - T_\delta(y)| < \delta\}|}{M} \approx 1.$$

We deduce (A) from the Marriage Lemma. Fix a standard $\delta > 0$ and for every $y \in Y$ set $S(y) = {}^*(F(y) - \delta, F(y) + \delta) \cap Y$. Let I be an arbitrary internal subset of Y . Set $S(I) = \bigcup_{y \in I} S(y)$ and $B(I) = \bigcup_{y \in I} {}^*(F(y) - \delta, F(y) + \delta)$. So, $S(I) = B(I) \cap Y$. The internal set $B(I)$ can be represented as a union of a hyperfinite family of disjoint intervals. Since the length of each of these intervals is not less than 2δ , their number is actually finite. Let $B(I) = \bigcup_{i=1}^n (\xi_i, \eta_i)$, where intervals (ξ_i, η_i) are pairwise disjoint and n is standard.

Consider the standard set $C = \bigcup_{i=1}^n (\circ \xi_i, \circ \eta_i)$. Then $dx(C)\mu_L(\text{st}^{-1}(*C))$. Obviously, $\text{st}^{-1}(*C)\Delta B(I) \subseteq \bigcup_{i=1}^n (M(\circ \xi_i) \cup M(\circ \eta_i)) = M(\partial C)$, where the monad of a number $a \in [0, 1]$ is denoted by $M(a)$. Since the Loeb measure of the monad of any number is equal to 0 and so, $M(\partial C) = 0$, one has $dx(C) = \circ \left(\frac{|S(I)|}{M} \right)$. Substituting $[0, 1]$ for X , dx for ν and τ for ψ in (27) obtain $dx(C) = dx(\tau^{-1}(C)) = \mu_L(F^{-1}(\text{st}^{-1}(C)))$. Since $I \setminus F^{-1}(\text{st}^{-1}(C)) \subseteq M(\partial C)$, one has $\circ \left(\frac{|I|}{M} \right) \leq \circ \left(\frac{|S(I)|}{M} \right)$. This means that if $r_I = \max\{0, |I| - |S(I)|\}$, then $\frac{r_I}{M} \approx 0$. Let $r = \max\{r_I \mid I \subseteq Y\}$. Fix an arbitrary set $Z \subseteq {}^*[0, 1] \setminus Y$ such that $|Z| = r$. For every $y \in Y$ set $S'(y) = S(y) \cup Z$ and for an arbitrary $I \subseteq Y$ set $S'(I) = \bigcup_{y \in I} S'(y)$. Then $S'(I) = S(I) \cup Z$, $|S'(I)| = |S(I)| + r \geq |I|$, since $|I| - |S(I)| = r_I \leq r$. By the Marriage Lemma there exists an injective map $\theta : Y \rightarrow S'(Y) = Y \cup Z$ such that $\forall y \theta(y) \in S'(y)$. Obviously $|\theta^{-1}(Z)| = |Y \setminus \theta(Y)| \leq r$. So, there exists a bijective map $\lambda : \theta^{-1}(Z) \rightarrow Y \setminus \theta(Y)$. Define $T_\delta : Y \rightarrow Y$ by the formula

$$T_\delta(y) = \begin{cases} \theta(y), & y \in Y \setminus \theta^{-1}(Z) \\ \lambda(y), & y \in \theta^{-1}(Z) \end{cases}$$

Notice that $\frac{|\theta^{-1}(Z)|}{M} \leq \frac{r}{M}$. By construction of T_δ one has $\forall y \in Y \setminus \theta^{-1}(Z) |T_\delta(y) - \tau(y)| < \delta$. Since $\mu_L(\theta^{-1}(Z)) \leq \frac{r}{M} \approx 0$, the statement (A) is proved.

Let $\mathcal{S}(Y)$ be the set of all internal permutations of Y . Consider the external function $f : \mathbb{N} \rightarrow \mathcal{S}(Y)$ such that $f(n) = T_{\frac{1}{n}}$. By the Saturation Principle the function f can be extended to an internal function $\tilde{f} : \{0, \dots, K\} \rightarrow \mathcal{S}(Y)$

for some $K \in {}^*\mathbb{N}_\infty$. Internal function $g(n) = \frac{|\{y \in Y \mid |F(y) - \tilde{f}(y)| \geq \frac{1}{n}\}|}{M}$ assumes only infinitesimal values for all standard n . By Robinson's Lemma there exists $L \in {}^*\mathbb{N}_\infty$ such that $g(L) \approx 0$. set $T = \tilde{f}(L)$. Then $\mu_L(\{y \in Y \mid T(y) \approx F(y)\}) = 1$. Since F is a lifting of τ , the same is true also for $T(y)$. This proves I.

We have to prove now that a h.a. T of τ can be chosen as a cycle of maximal length.

II. Fix a permutation $T : Y \rightarrow Y$ that is a h.a. of τ and represent it by a product of pairwise disjoint cycles, including the cycles of length 1 (fix points):

$$T = (y_{11} \dots y_{1n_1})(y_{21} \dots y_{2n_2}) \dots (y_{b1} \dots y_{bn_b}), \quad (28)$$

where $y_{ij} \in Y$ is the j -th element in the i -th cycle and b is the number of cycles. So,

$$\sum_{i=1}^b n_i = M = |Y|. \quad (29)$$

We assume also that $n_1 \geq n_2 \geq \dots \geq n_b$. Consider the cycle

$$C = (y_{11} \dots y_{1n_1} y_{21} \dots y_{2n_2} \dots y_{b1} \dots y_{bn_b}) \quad (30)$$

By (29) C is a cycle of length M , i.e. a transitive permutation.

Set $B = \{y \in Y \mid C(y) \neq T(y)\}$.

$$|B| = b = \sum_{n=1}^M a_n, \quad (31)$$

where a_n is the number of cycles of length n .

III. Recall that a point $x \in [0, 1]$ is said to be an n -periodic point of the transformation τ if its orbit under this transformation consists of n -points: $x, \tau x, \dots, \tau^{n-1}x$. A point x is said to be τ -periodic if it is n -periodic for some n . The transformation τ is said to be aperiodic if the set of periodic points has measure zero. It is well-known that every measure preserving automorphism τ of a Lebesgue space X defines the partition of this space by τ -invariant Lebesgue subspaces of aperiodic and n -periodic points. So, it is enough to prove our statement for the case of aperiodic transformation τ and for the case of n -periodic transformation τ .

Suppose that the transformation τ is aperiodic. Let us prove that under this assumption the cycle C defined in the part II is a h.a. of τ .

Let $P_n(T) \subseteq Y$ be the set of all n -periodic points of T and let $P_n(\tau) \subseteq X$ be the set of all n -periodic points of τ . Since T is a lifting of τ it is easy to see that for every standard k the following relations

$$T(y) \approx \tau(\circ y), \dots, T^k(y) \approx \tau^k(\circ y) \quad (32)$$

hold μ_L -a.e. on Y . So, for every standard n $P_n(T) \subseteq st^{-1}(P_n(\tau))$ up to a set of the Loeb measure zero. Since $dx(P_n(\tau)) = 0$, one has $\frac{1}{M}|P_n(T)| \approx 0$. Obviously, $|P_n(T)| = na_n$. Thus, for every standard n one has $\frac{1}{M} \cdot a_n \approx 0$. By the Robinson's Lemma there exists an infinite N such that $\frac{1}{M} \sum_{n=1}^N a_n \approx 0$. Obviously $M \geq \sum_{n=N+1}^M a_n \cdot n \geq (N+1) \sum_{n=N+1}^M a_n$. So, $\frac{1}{M} \sum_{n=N+1}^M a_n \leq \frac{1}{N+1} \approx 0$ and $\frac{1}{M} \cdot |B| = \frac{1}{M} \sum_{n=1}^M a_n \approx 0$. Thus, $\mu_L(B) = 0$, $C(y) = T(y)$ μ_L -a.e. and C approximates τ .

IV. Suppose now that τ is n -periodic. We prove first that a h.a. T of τ also can be chosen to be n -periodic. The relations (32) imply that for almost every point $y \in Y$ if y has a standard period with respect to T , then this period is a multiple of n . Indeed, if y satisfies (32), and its standard period is $nq + r$ for $0 < r < n$, then ${}^\circ y = {}^\circ T^{nq+r}(y) = \tau^{nq+r}({}^\circ y) = \tau^r({}^\circ y)$, which is impossible since τ is n -periodic. By Saturation Principle, there exist an internal set $I \subseteq Y$ such that $\mu_L(I) = 1$ and a number $N \in {}^*\mathbb{N}_\infty$ such that for every point $y \in I$, whose period is less, than N , this period is a multiple of n .

Consider the representation (28) of T and set $n_i = nq_i + r_i$, $r_i < n$ for each $i \leq b$. Let $Y' \subseteq Y$ be the set obtained by deleting from Y the last r_i elements of the i -th cycle for each $i \leq b$. The set Y' has the Loeb measure equal to 1. Indeed, all the deleted elements either belong to the set $Y \setminus I$, whose measure is 0, or to a cycle whose length is greater, than N . The number of these cycles does not exceed $\frac{M}{N}$ and the number of deleted points in each such cycle is less, than n . So the Loeb measure of the set of these points is also equal to 0. Since $\mu_L(Y') = 1$ the pair (Y', st) is a h.a. of $[0, 1]$. The construction of Y' defines also the permutation $T' : Y' \rightarrow Y'$ such that

$$T' = (y_{11} \dots y_{1n-q_1})(y_{21} \dots y_{2n-q_2}) \dots (y_{b1} \dots y_{bn-q_b}). \quad (33)$$

Notice, that actually the number of cycles in T' may be less, than b , since in case of $q_i = 0$ the i -th cycle is empty. However, the dynamical system (Y', μ_L, T') is a h.a. of the dynamical system (X, v, τ) . Indeed, let $D = \{y \in Y' \mid T(y) \neq T'(y)\}$. Then $D \subseteq \{y \in Y' \mid T(y) \in Y \setminus Y'\} \subseteq T^{-1}(Y \setminus Y')$. Thus, $\mu_L(D) \leq \mu_L(Y \setminus Y') = 0$. To obtain an n -periodic h.a. of τ it is enough to split each cycle in the representation (33) in cycles of length n . Indeed, let the obtained cycle be

$$T'' = (z_1, \dots, z_n)(z_{n+1}, \dots, z_{2n}) \dots (z_{(i-1)n+1}, \dots, z_{in}) \dots (z_{(K-1)n+1}, \dots, z_{Kn}),$$

where $K = |Y'|/n$. It is easy to see that $T''(y) \neq T'(y)$, only for the points z_{in} . Notice, that $\mu_L(\{z_{in} \mid i \leq K\}) = \frac{1}{n} > 0$. However, due to (32) and the n -periodicity of τ , for almost all of these points one has

$$T'(z_{in}) = z_{in+1} \approx \tau({}^\circ z_{in+1}) = \tau^n({}^\circ z_{(i-1)n+1}) = {}^\circ z_{(i-1)n+1}.$$

At the same time $T''(z_{in}) = z_{(i-1)n+1}$ by the definition. Thus, $T''(y) \approx T'(y)$ for almost all y .

V. To complete the proof of the theorem for $X = [0, 1]$ we need to consider the case when all orbits of T have the same standard period n . In this case $M = N \cdot n$.

It is easy to see that there exists a selector $I \subset Y$ (subset that intersect each orbit of T by a single point) that is dense in ${}^*[0, 1]$, i.e. the monad $M(I) = {}^*[0, 1]$. It is enough to show the existence of a selector that intersects every interval with rational endpoints. Obviously, for every finite set A of such intervals, there exists a selector that intersects each interval from A . The existence of a dense selector follows from the Saturation Principle.

Let $I = \{y_1 < y_2 < \dots < y_N\}$ be a dense selector. Here $<$ is the order in ${}^*[0, 1]$. Due to the density of I in ${}^*[0, 1]$ for every $k < N$ one has $y_k \approx y_{k+1}$. Obviously, the transformation T can be represented by a product of pairwise disjoint cycles as follows:

$$T = (y_1, \dots, T^{n-1}y_1)(y_2, \dots, T^{n-1}y_2) \dots (y_N, \dots, T^{n-1}y_N)$$

Consider the following cycle S of the length M :

$$S = (y_1, \dots, T^{n-1}y_1 y_2, \dots, T^{n-1}y_2 \dots y_N, \dots, T^{n-1}y_N)$$

Since for every $k \leq N$ holds $T^n(y_k) = y_k$, one has

$${}^\circ S(T^{n-1}(y_k)) = {}^\circ y_{k+1} = {}^\circ y_k = {}^\circ T^n(y_k) = {}^\circ T(T^{n-1}(y_k)) = \tau({}^\circ T^{n-1}(y_k))$$

for almost all k . Thus, ${}^\circ S(y) = \tau({}^\circ y)$ for almost all y and the cycle S is a h.a. of τ .

We proved actually that for every h.a. Y of $([0, 1], dx)$ there exists an internal set $Y' \subseteq Y$ with $\mu_L(Y') = 1$ and a permutation $T' : Y' \rightarrow Y'$ such that the hyperfinite dynamical system (Y', μ_L, T') is a h.a. of the dynamical system

(X, ν, τ) and T' is a transitive permutation of Y' (see Part IV of this proof). To obtain a transitive h.a. $T : Y \rightarrow Y$ of τ , set $T' = (z_1, \dots, z_{|Y'|})$ and $Y \setminus Y' = \{u_1, \dots, u_{|Y \setminus Y'|}\}$ and consider the cycle of the length $|Y|$

$$T : (z_1, \dots, z_{|Y'|}, u_1, \dots, u_{|Y \setminus Y'|})$$

Since $\mu_L(\{y \in Y \mid T'(y) \neq T(y)\}) = 0$ the transformation T is h.a. of τ .

VI. The statement of the theorem for the case of an arbitrary dynamical system (X, ν, τ) , satisfying the conditions, follows immediately from Lemma 2. Indeed, let a set $B \subseteq X$, a set $C \subseteq [0, 1]$, a bijective map $\psi : B \rightarrow C$ and a bijective lifting $G : Y \rightarrow {}^*[0, 1]$ of ψ satisfy the conditions of Lemma 2. Then $\lambda = \psi\tau\psi^{-1} : [0, 1] \rightarrow [0, 1]$ is a measure preserving transformation. Fix an arbitrary h.a. Y of the measure space (X, ν) . Then by Lemma 2 the hyperfinite set $Z = G(Y)$ is a h.a. of $([0, 1], dx)$. By the results proved in the parts I-V, there exists a permutation $S : Z \rightarrow Z$ that is a h.a. of λ . Then it is easy to see that the permutation $T = G^{-1}SG : Y \rightarrow Y$ is a h.a. of τ . Obviously, if S is a transitive permutation, then T is a transitive permutation as well. \square

iii) (*Proof of Theorem 9.*) Let (X, ρ) be a compact metric space. Consider a hyperfinite set $Y \subseteq {}^*X$. This set defines a Borel measure ν_Y on X by the formula $\nu_Y(K) = \mu_L(st^{-1}(K) \cap Y)$. Obviously Y is a h.a. of the measure space (X, ν_Y) . Let $T : Y \rightarrow Y$ be an internal permutation that is S -continuous on A for some (not necessary internal) set $A \subseteq Y$ with $\mu_L(A) = 1$, i.e.

$$\forall a_1, a_2 \in A \ (a_1 \approx a_2 \implies T(a_1) \approx T(a_2)). \quad (34)$$

Notice that since $st^{-1}(st(A)) \supseteq A$ and $\mu_L(A) = 1$, the set $st(A) \subseteq X$ is a measurable set w.r.t. the completion of the measure ν_Y , which we denote by ν_Y also, and $\nu_Y(st(A)) = 1$.

Define a map $\tau_T : X \rightarrow X$ such that $\tau_T(st(y)) = st(T(y))$ for $y \in A$ and $\tau_T \upharpoonright X \setminus st(A)$ is an arbitrary measurable permutation of the set $X \setminus st(A)$.

Proposition 21 *The map τ_T preserves the measure ν_Y .*

Proof. Replacing, if necessary, A by $\bigcap_{n \in \mathbb{N}} T^n(A)$ we may assume that A is invariant for permutation T . Then, obviously, $st(A)$ is invariant for τ_T .

Consider a closed set $B \subseteq X$. We have to prove that $\nu_Y(\tau_T^{-1}(B)) = \nu_Y(B)$. One has

$$\nu_Y(\tau_T^{-1}(B)) = \nu_Y(\tau_T^{-1}(B) \cap st(A)).$$

It is easy to check that

$$\tau_T^{-1}(B) \cap st(A) = st(T^{-1}(st^{-1}(B)) \cap A).$$

Thus,

$$\nu_Y(\tau_T^{-1}(B)) = \mu_L(st^{-1}(st(T^{-1}(st^{-1}(B)) \cap A))) = \mu_L((st^{-1}(st(T^{-1}(st^{-1}(B)) \cap A))) \cap A)$$

Using (34) and the T -invariance of it is easy to check, that

$$st^{-1}(st(T^{-1}(st^{-1}(B)) \cap A)) \cap A = T^{-1}(st^{-1}(B)) \cap A.$$

So,

$$\nu_Y(\tau_T^{-1}(B)) = \mu_L(T^{-1}(st^{-1}(B)) \cap A) = \mu_L(T^{-1}(st^{-1}(B))) = \mu_L(st^{-1}(B)) = \nu_Y(B).$$

In the last chain of equalities we used the facts that $\mu_L(A) = 1$ and that T being a permutation preserves the Loeb measure. \square

Proposition 22 1) *In conditions of Proposition 21 for any $a > 0$ and for any $y \in Y$ the following positive functional $l_a(\cdot, T, y)$ on $C(X)$ is defined: $l_a(f, T, y) = {}^\circ A_K({}^*f, T, y)$, where ${}^\circ(\frac{K}{M}) = a$ and $f \in C(X)$*

2) *If $\forall K, L \in {}^*\mathbb{N}_\infty$ ($\frac{K}{M} \approx \frac{L}{M} \approx 0 \implies A_K({}^*f, T, y) \approx A_L({}^*f, T, y)$), then $l_0(f, T, y)$ is defined by the same formula as in 1). In this case $l_0(f, T, y) = \tilde{f}(y)$*

3) *If $T : Y \rightarrow Y$ is S -continuous, then the functional $l_0(\cdot, T, y)$ is τ_T -invariant for all $y \in Y$.*

Proof. The correctness of the definition in 1) follows from Theorem 6. The statement 2) follows from Proposition 20. To prove statement 3) notice that if T is S -continuous on Y , then τ_T is continuous on X and, thus, ${}^*(f \circ \tau_T) \upharpoonright Y$ is a lifting of $f \circ \tau_T$. So,

$$\forall y \forall K \in {}^*\mathbb{N} \quad {}^*(f \circ \tau_T)(T^K(y)) \approx f(\tau_T({}^\circ T^K(y))) = f({}^\circ T^{K+1}(y)) \approx {}^*f(T^{K+1}y)$$

These equivalences allows to prove that $A_K({}^*(f \circ \tau), T, y) \approx A_K({}^*f, T, y)$. \square

Now we can complete the proof of Theorem 9

Let $y \in Y$ satisfy conditions of the theorem. For a number $K \in {}^*\mathbb{N}$ we denote the initial K -segment of the T -orbit of Y by $S(K, y)$. Then for any $K \in {}^*\mathbb{N}_\infty$ one has $\text{st}(S(K, y)) = Y$, since the closed set $\text{st}(S(K, y))$ contains the τ -orbit of $\text{st}(y)$. Let K be a T -period of y . Then $K \in {}^*\mathbb{N}_\infty$. Otherwise, the τ -orbit of $\text{st}(y)$ would be finite, while we assume X to be infinite. It is easy to see that it is enough to prove the theorem for every $N \in {}^*\mathbb{N}_\infty$ such that $N \leq K$. Under this assumption all elements of the set $Y_1 = \{y, Ty, \dots, T^{N-1}y\}$ are distinct. Since $\text{st}(Y_1) = X$, the set Y_1 defines the Borel measure ν_{Y_1} on X . Let $T_1 : Y_1 \rightarrow Y_1$ be the permutation of Y_1 that differs from T only for one element $T^{N-1}y$: $T_1(T^{N-1}y) = y$. Set $A = Y_1 \setminus \{T^{N-1}y\}$. Then X, τ, Y_1, T_1 , and A satisfy conditions of Proposition 21: $\mu_L(A) = 1$, $\forall z \in A \text{ st}(T_1 z) = \tau(\text{st}(z))$, i.e. $\tau_{T_1} = \tau$ and T_1 is S -continuous on A , since τ is a continuous map. By Proposition 21 the measure ν_{Y_1} is τ -invariant. Thus, $\nu_{Y_1} = \nu$ due to the unique ergodicity of the map τ . If $f \in C(X)$, then obviously ${}^*f \upharpoonright Y_1$ is an S -integrable lifting of f . This proves the equality (16). \square

References

- [1] S. Albeverio, J.E. Fenstad, R. Hoeg-Krohn, T. Lindstrom, *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press, Orlando, 1986.
- [2] M. Brin, G. Stuck, *Introduction to Dynamical Systems*, Cambridge University Press, 2002.
- [3] I.P. Cornfeld, S.V. Fomin, and Ya.G. Sinai, *Ergodic Theory* Springer-Verlag, New York, 1982.
- [4] N.G. de Bruijn, A combinatorial problem, *Koninklijke Nederlandse Akademie Wetenschappen*, 1946, **49**, pp. 758 - 764.
- [5] T. van Aardenne-Ehrenfest, N.G. de Bruijn, Circuits and trees in oriented linear graphs, *Bull. Belgium Math. Soc. - Simon Stevin*, 1951, **28**, pp. 203 - 217.
- [6] A.M. Alhakim, A Simple Combinatorial Algorithm for de Bruijn sequences, *American Math. Monthly*, 2010, **117**, pp. 728 - 732.
- [7] E.I. Gordon, *Nonstandard Methods in Commutative Harmonic Analysis*, American Mathematical Society, Providence, RI, 1977.
- [8] E.I. Gordon, C.W. Henson, P.A. Loeb. On representation of dynamical systems on Lebesgue spaces by hyperfinite dynamical systems. *Abstracts of Papers Presented to the AMS*, 2004, **25**, p. 62.
- [9] E.I. Gordon, A.G. Kusraev and S.S. Kutateladze, *Infinitesimal Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 2002.
- [10] V. Kanovei, M. Reeken, *Nonstandard Analysis, Axiomatically*, Springer-Verlag, Berlin-Heidelberg, 2004
- [11] P.A. Loeb and M.P.H. Wolff, *Nonstandard Analysis for the Working Mathematician*, Kluwer Academic Publishers, Dordrecht-Boston-London, 2000.
- [12] E. Nelson, *Radically Elementary Probability Theory*, Annals of Mathematics Studies, Princeton University Press, 1987.
- [13] E. Nelson, Internal set theory: a new approach to nonstandard analysis, *Bull. Amer. Math. Soc.* 1977, **83**, pp. 1165 - 1198.

- [14] . D. Zeilberger. (2001) Real Analysis is a Degenerate Case of Discrete Analysis. *New Progress in Difference Equations*, edited by Bernd Aulbach, Saber Elyadi and Gerry Ladas, (Proc. ICDEA 2001), Bernd Aulbach, ed., Taylor and Frances London (<http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/real.html>).

Gordon at Eastern Illinois University 600 Lincoln Avenue Charleston, IL 61920-3099 USA, email: cfyig@eiu.edu.

Glebsky at IICO-UASLP AvKarakorum 1470 Lomas 4ta Session SanLuis Potosi SLP 7820 Mexico, email: glebsky@cactus.iico.uaslp.mx

Henson at University of Illinois at Urbana-Champaign 1409 W. Green Street Urbana, Illinois 61801-2975 USA, email: henson@math.iuc.edu